Pricing catastrophe options in incomplete market

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Abstract

In complete markets, pricing financial products is easy (at least from a theoretical point of view). In incomplete markets (e.g. when the underlying process has jumps with random size, such has an insurance loss process), the price is no longer unique. So on the one hand, it becomes difficult to provide a tractable price of insurance-linked derivatives. On the other hand, when facing catastrophic losses, using the pure premium as a price might not be relevant (e.g. for solvency issues). Both financial market and (re)insurance industry have proposed techniques to price identical hedging products that can be related (e.g. Esscher transform and more generally distorted risk measures in insurance, Gerber-Shiu transform in finance). In this paper, we focus on indifference utility techniques, assuming that stock prices have jumps, related to major catastrophic losses, and thus, partial hedging should then be possible.

1. Introduction and motivations

1.1. Notations and definition

The buyer of an insurance contract buys the right to get reimbursed - by the insurance company - all the losses which occurred during a given period of time, (for which the loss amount exceeded a deductible, if any). The buyer of a call options buys the right to buy the underlying stock from the seller to capture its increased value above the strike price.

Both (financial) options and insurance policies have the objective to transfer a risk from one part to another, against a specific payment (called premium in insurance). But classical techniques in insurance (based on the use of the pure premium, $\mathbb{E}_{\mathbb{P}}((X-d)_+)$) and finance (based on the assumption of complete market and no-arbitrage, so that the price of a call option is $\mathbb{E}_{\mathbb{Q}}((X-K)_+)$) are no longer relevant.

On the one hand, most of the techniques designed to price insurance contracts have been developed for standard risks, not to hedge against catastrophes. Pricing reinsurance, where events are rare and with high severity, is more challenging, and the use is the pure premium might not be relevant, for solvency issues. On the other hand, the closed-form model for pricing financial options obtained in the beginning of the 70's, assumed that a volatility of the underlying stock was available, known and constant, and that the underlying price was continuous. Those two assumption assumptions were related to the idea of *complete* markets.

The challenge in insurance-linked derivatives is find a price for those financial products, and to relate them to classical insurance covers, since question asked by any risk manager is "*which risk transfer technique is the cheapest one* ?". But, as mentioned in Finn and Lane (1995), one has to keep in mind, "*there are no right price of insurance, there is simply the transacted market price which is high enough to bring forth sellers and low enough to induce buyers*". From a terminology point of view, Holtan (2007) suggested to distinguish the *price* of an option or the *premium* of an insurance contract, and the so-called *value* of those products. The difference depending mostly on market conditions.

1.2. Trading insurance risks

Insurance risks are traded as long as there are insurance contracts, buyers and sellers, but they are traded within the (re)insurance market *only*. To compare with the financial market, derivatives are traded on structured market, as well as the underlying stock, which will make replication possible (and therefore hedging and pricing derivatives). In the case of insurance risks, we can image that some standard contracts could be - somehow - concluded with financial companies, but the underlying risk (cumulated insurance claims for indemnity covers, or weather related index) is not traded on financial market: in that case, there is few chance that insurance risks could be replicated, and therefore classical techniques to price are no longer valid.

Assuming that financial markets integrate information about catastrophes (are more generally any insurance related information), it might be possible to hedge insurance risks on financial markets. But most of the assumptions underlying the Black & Scholes assumptions are usually not fulfilled with insurance-linked derivatives

- the market is not complete, and catastrophe (or mortality risk) cannot be replicated,
- the guarantees are not actively traded, and thus, it is difficult to assume no-arbitrage
- the hedging portfolio should be continuously rebalanced, and there should be large transaction costs
- if the portfolio is not continuously rebalanced, we introduce an hedging error
- equities prices are not driven by a geometric Brownian motion process

The goal of this paper is to focus on catastrophe options and to find *a* price for those financial products.

1.3. Outline of the paper

In Section 2, classical results on financial pricing will be recalled, focusing on assumptions underlying the *fundamental theorem of asset pricing*. Then, in Section 3, classical insurance pricing methods will be presented (based either on expected utility principles or using distorted risk measures). In Section 4, classical financial method to avoid drawbacks of uncompleteness will be presented (and related to insurance pricing), to find *a* possible martingale measure.

And finally, in Section 5, we will study a model based on indifference utility pricing. The underlying idea is that financial markets can be affected by shocks related to major insurance losses. As mentioned in Shimpi (1995) with a qualitative point of view, "*from an insurance industry perspective, the closer the index is to the loss experience, the better the ability to hedge the loss exposure of insurers*". Even, if a stock price is not perfectly correlated with insurance losses, at least its discontinuous part can be. The goal here will be to see if those jumps in financial prices can be used to hedge again catastrophes.

2. Pricing financial products in complete markets

Harrison and Pliska (1981) said that a market is *complete* if there is only one equivalent martingale measure to the underlying stock price. Insurance markets would be complete if the would be a unique price for each risk, and if each contract could perfectly be hedged in the market. As mentioned in Embrechts and Meister (1997) market uncompleteness can be explained by jumps in the underlying stochastic process, with random size, by stochastic volatility, or by the existence of transaction costs (or more generally any *friction*). Hence, in complete markets, all relevant market information is supposed to be known and integrated in the price: no investor will expect a higher return than the risk free rate of return. The technique is to tune the historical probability $\mathbb{P}$ into an equivalent probability measure $\mathbb{Q}$, so that the price process of the underlying financial asset becomes a martingale under probability $\mathbb{Q}$, i.e. $\mathbb{E}_{\mathbb{Q}}(S_{t+h}|\mathcal{F}_t) = S_t$. Hence, it becomes impossible to use history of the stock to earn money: all possible relevant information is already included in the its spot price. This link between no-arbitrage assumption and martingale processes is the *fundamental theorem of asset pricing* (see Delbaen and Schachermayer (1994)): the price of a contingent claim X (e.g. the payoff of the European call with strike K and maturity T is $X = (S_T - K)_+$) is $\pi(X) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X)$, assuming constant risk free rate r, where $\mathbb{Q}$ stands for *the* risk neutral probability measure equivalent to $\mathbb{P}$.

The Black & Scholes model assumes that the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$ where $(X_t)_{t\geq 0}$ is a Brownian process, i.e. $S_t = S_0 \exp(\mu + \sigma X_t^0)$ where $(X_t^0)_{t\geq 0}$ is a standard Brownian motion. Having a geometric Brownian motion reflecting uncertainty on financial markets (for stock prices (S_t)) yield simple and nice pricing formulas. The most difficult practical issue is that the only unknown valuation parameter is the stock volatility σ, making option dealers simply "*volatility dealers*": the value of the a financial option depends on the volatility of the underlying financial stock, and not its expected return (which has to be equal to the risk free rate of return), leading to a "*risk neutral*" pricing.

3. Pricing insurance products

The basic principle of insurance is the law of large numbers: if the premium asked is $\mathbb{E}_{\mathbb{P}}(X)$, then the insurance company makes a null profit, *on average*. Feller (1945) called $\mathbb{E}_{\mathbb{P}}(X)$ the *fair* price (of a game, in his terminology). In the terms of d'Alembert, the pure premium is the "*inner product of probabilities and losses*". Thus, $\mathbb{E}_{\mathbb{P}}(X)$ is called pure premium, but using it as the price of a risk, the company is very likely to lose money (since the balance is only *on average*. Therefore, traditional premium calculation principles are

$$\begin{cases} \pi(X) = \mathbb{E}_{\mathbb{P}}(X)\text{: equivalence principle (pure premium)} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda \mathbb{E}_{\mathbb{P}}(X)\text{: expected value principle} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda Var_{\mathbb{P}}(X)\text{: variance principle} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda \sqrt{Var_{\mathbb{P}}(X)}\text{: standard-deviation principle} \end{cases}$$

Remark 3.1 *For the standard-deviation principle, if X has a Gaussian distribution, then $\pi(X)$ is simply a quantile of X.*

3.1. Pricing using expected utility principles

The fact that the pure premium might not be appropriate has been mention starting from Saint-Petersburg's paradox. One of the answer was to introduce a *moral utility* of X. A utility function U is an increasing twice differentiable function on $\mathbb{R}$, strictly increasing ($U'(\cdot) > 0$, i.e. "*more is better*") and concave ($U''(\cdot) < 0$, i.e. "*marginal utility decreases*"). Concavity is related to risk aversion; since we assume that the agent is willing to transfer a risk, it is relevant to assume that U is concave.

Example 3.1 *Three types of expected utility are frequently used in the context of expected utility,*

$$\begin{cases} \forall x \in \mathbb{R}_+^\star, U_L(x) = \log(x)\textit{: logarithmic utility} \\ \forall x \in \mathbb{R}_+^\star, U_P(x) = \frac{x^p}{p} \textit{ where } p \in]-\infty, 0[\cup]0, 1[\textit{: power utility} \\ \forall x \in \mathbb{R}, U_E(x) = -\exp(-\frac{x}{x_0})\textit{: exponential utility}. \end{cases}$$

Functions U_P and U_L belong to the set of functions have constant relative risk aversion (CRRA). Functions U_E belong to the set of functions have constant absolute risk aversion (CARA).

Given utility function U, the premium π that an agent is willing to pay to transfer loss X is a solution of the following equation

$$U(\omega - \pi) = \mathbb{E}_{\mathbb{P}}(U(\omega - X)) \tag{1}$$

where ω denotes initial wealth of the insured. Using Jensen's inequality (since U is assumed to be concave), note that $\pi \geq \mathbb{E}_{\mathbb{P}}(X)$.

Example 3.2 *Assuming exponential utility, i.e. $U(x) = -e^{-x/x_0}$ (with constant risk aversion $1/x_0$), then $\pi = x_0 \log \mathbb{E}_{\mathbb{P}}(e^{X/x_0})$ (also called entropy measure).*

Borch (1962) observed that the price of reinsurance contracts obtained with HARA utility functions (more general than CARA and CRRA) is quite similar to the financial formulas: "*This indicates that the theory of insurance premiums and the theory of asset prices are special cases of a more general theory*". This emphasizes the idea that it should be possible to relate insurance and finance valuation techniques.

3.2. Pricing using distorted risk measures

Using the duality principle (see Yaari (1987)), instead of distorting losses using a utility function, an alternative is to use a distortion of probabilities (leading to the *dual* approach, since the expected value can be seen as an inner product, as mentioned already by d'Alembert). Hence, the agent solve the dual version of Equation (1), i.e. (with an abuse of notation to highlight duality, see Remark 3.2)

$$\omega - \pi = \mathbb{E}_{g\circ\mathbb{P}}(\omega - X) = \int (\omega - x) g \circ \mathbb{P}(dx), \tag{2}$$

or equivalently, $\pi = \displaystyle\int x g \circ \mathbb{P}(dx) = \int g(\mathbb{P}(X > x))dx$ in the case X is a positive random variable, where g is a *distortion* measure, i.e. an increasing function on $[0, 1]$, with $g(0) = 0$ and $g(1) = 1$.

Remark 3.2 *Note that this probability distortion does not necessarily define a probability measure, but only a* capacity*: if $\mathbb{Q} = g \circ \mathbb{P}$, $\mathbb{Q}(\varnothing) = 0$ (since $g(0) = 0$), $\mathbb{Q}(\Omega) = 1$ (since $g(1) = 1$), and $\mathbb{Q}(A) \leq \mathbb{Q}(B)$ if $A \subset B$ (since g is an increasing function). Hence, in Equation (2) $\mathbb{E}_{g\circ\mathbb{P}}$ is not an expected value, but a Choquet integral with respect to (nonadditive) measure $g \circ \mathbb{P}$.*

Example 3.3 *If $g(x) = \mathbf{1}(x > \alpha)$, then $\pi = F^{-1}(1 - \alpha)$, $\alpha \in (0, 1)$ and $F(x) = \mathbb{P}(X \leq x)$.*

As a particular case of distorted probabilities, an important principle is the use of the Esscher transform,

$$\pi = \mathbb{E}_{\mathbb{Q}}(X) = \frac{\mathbb{E}_{\mathbb{P}}(X \cdot e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})},$$

for some $\alpha > 0$. More generally, Delbaen and Haezendonck (1989) considered the following change of measure, so that cumulative distribution function of the Radon-Nikodym derivative $d\mathbb{P}/d\mathbb{Q}$ is

$$G(x) = \frac{1}{\mathbb{E}_{\mathbb{P}}(e^{\beta(X)})} \int_0^x \exp(\beta(y))dF(y), x \geq 0,$$

where F is the distribution function of X under $\mathbb{P}$, and $\beta(\cdot) : [0, \infty) \rightarrow (-\infty, \infty)$ satisfies $\mathbb{E}_{\mathbb{P}}(e^{\beta(X)}) < \infty$ and $\mathbb{E}_{\mathbb{P}}(Xe^{\beta(X)}) < \infty$.

Example 3.4 *If $\beta(x) = \log(1 + b(x - \mathbb{E}_{\mathbb{P}}(X)))$, then $\pi = \mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(X) + bVar_{\mathbb{P}}(X)$, which is the variance principle. If $\beta(x) = \alpha x - \log \mathbb{E}_{\mathbb{P}}(e^{\alpha X})$, for some $\alpha > 0$, then $\pi = \mathbb{E}_{\mathbb{Q}}(X) = \dfrac{\mathbb{E}_{\mathbb{P}}(X \cdot e\alpha X)}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})}$.*

4. Pricing financial products in incomplete markets

4.1. A *natural* framework based on Lévy processes

As mentioned in Section 2, market uncompleteness arises when the underlying stochastic process has jumps with random size. Hence, in a general framework, assume that the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$ where $(X_t)_{t\geq 0}$ is a Lévy process. Recall that $(X_t)_{t\geq 0}$ has independent, infinitely divisible and stationary increments, thus $X_{t+h} - X_t$ has characteristic function ϕ^h. The cumulant characteristic function satisfies the Lévy-Khintchine formula, i.e.

$$\psi(u) = \log \phi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is the so-called Lévy measure on $\mathbb{R}/\{0\}$. Hence, the Lévy process is characterized either by ϕ (the characteristic function of X_1), or by the triplet (γ, σ^2, ν) in the Lévy-Khintchine formula.

Remark 4.1 *Again, except the case when $(X_t)_{t\geq 0}$ is a (pure) Poisson process or a Brownian motion, any Lévy model is an incomplete model.*

Market completeness is related to the existence of a *unique* martingale measure, also called the *predictable representation property* of a martingale: a martingale $(M_t)_{t\geq 0}$ satisfies this property if and only if for any square-integrable random variable $Z \in \mathcal{F}_T$, there exists a $\mathcal{F}_t$-predictable process $(a_t)_{t\in[0,T]}$ such that $Z = \mathbb{E}(Z) + \int_0^T a_s dM_s$. Actually, $(a_t)_{t\in[0,T]}$ is related to the self-balancing strategy. Nualart and Schoutens (2000) proved that under some weak assumptions, a Lévy process $(X_t)_{t\geq 0}$ can also have a *predictable representation property* of the form

$$Z = \mathbb{E}(Z) + \sum_{i=1}^{\infty} \int_0^T a_s^{(i)} d(H_s^{(i)} - \mathbb{E}(H_s^{(i)})),$$

where the $(a_t^{(i)})_{t\in[0,T]}$'s are $\mathcal{F}_t$-predictable processes, and $H_t^{(i)} = \sum_{0<s\leq t}[X_s - X_{s^-}]^i$, where times s are times where the Lévy process jumps. As mentioned in Schoutens (2003), the predicable integrands $(a_t^{(i)})_{t\in[0,T]}$'s appearing in this representation can be interpreted in terms of minimal variance strategies. Hence, those processes correspond to the risk that cannot be hedged away. The term $(a_t^{(1)})_{t\geq 0}$ leads the strategy that realizes the *closest* hedge to the claim.

A first idea, related to the classical pricing process in complete market is to find an equivalent martingale measure, and to use it to derive a *price*. Hence, in Section 4.2 we will mention two ideas widely used to obtain one equivalent martingale measure $\mathbb{Q}$: one based on Gerber and Shiu (1994) (i.e. Esscher transform from insurance pricing) and the other one based on some *mean-correcting martingale measure*. The main problem in incomplete market is that there is no replication portfolio. But it is still possible to super-replicate.

4.2. Finding one risk neutral measure

4.2.1. Using the Esscher transform

Following Gerber and Shiu (1994) we can - by using the Esscher transform - find in some cases at least one equivalent martingale measure $\mathbb{Q}$. More generally, Bühlmann *et al.* (1998) discussed the Esscher transform for specific classes of semi-martingales, with applications in finance and insurance.

Given a Lévy process $(X_t)_{t\geq 0}$ under $\mathbb{P}$ with characteristic function ϕ or triplet (γ, σ^2, ν), then under Esscher transform probability measure $\mathbb{Q}_\alpha$ (as defined in Section 3.2), $(X_t)_{t\geq 0}$ is still a Lévy process with characteristic function ϕ_α such that

$$\log \phi_\alpha(u) = \log \phi(u - i\alpha) - \log \phi(-i\alpha),$$

and triplet $(\gamma_\alpha, \sigma^2_\alpha, \nu_\alpha)$ for X_1, where $\sigma^2_\alpha = \sigma^2$, and

$$\gamma_\alpha = \gamma + \sigma^2\alpha + \int_{-1}^{+1} (e^{\alpha x} - 1)\nu(dx) \text{ and } \nu_\alpha(dx) = e^{\alpha x}\nu(dx),$$

see e.g. Schoutens (2003).

Example 4.1 *A particular case is given when $(X_t)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}$, then if $\alpha = (r - \mu)/\sigma^2$, $(X_t)_{t\geq 0}$ is still a Brownian motion under $\mathbb{Q}_\alpha$.*

Proposition 4.2 *If the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$, where $(X_t)_{t\geq 0}$ is a Lévy process, such that random variable X_1 is non-degenerate and possesses a moment generating function $M(t) = \mathbb{E}_{\mathbb{P}}(e^{tX})$ on some interval (a, b), and if there exists $u \in (a, b - 1)$ such that $M(1 + u) = M(u)$, then $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}_u$-martingale.*

Proof. Shiryaev (1999) ■

In order to have unicity, additional assumptions are necessary (see Kallsen and Shiryaev (2002)).

4.2.2. A mean-correcting martingale measure

Another way to obtain an equivalent martingale measure is inspired from the Black & Scholes model, and is related to some *mean-correcting martingale measure*. The underlying idea is to note that given a Lévy process $(X_t)_{t\geq 0}$ under $\mathbb{P}$ with characteristic function ϕ and triplet (γ, σ^2, ν), then the shifted process $(Y_t)_{t\geq 0} = (X_t - mt)_{t\geq 0}$ is also a Lévy process with characteristic function $\phi_m(u) = e^{ium}\phi(u)$ and triplet $(\gamma_m, \sigma^2_m, \nu_m) = (\gamma + m, \sigma^2, \nu)$ for X_1, see e.g. Schoutens (2003).

In the Black and Scholes model, we just switch from mean $\mu - \sigma^2/2$ to $r - \sigma^2/2$. In the Lévy model, the idea is to use the same kind of transform, $m_{\text{new}} = m_{\text{old}} + r - \log \phi(-i)$ (in the Black and Scholes model, $\log \phi(-i) = \alpha$). The choice of m_{new} will be such that the discounted price is a martingale.

5. IThe indifference utility approach

As point out in Swiss Re (1999), about the pricing of financial stop loss contracts "*the risk-neutral valuation technique traditionally used for the pricing of financial derivatives cannot be applied directly*". Nevertheless, practitioners need *a* price for insurance-linked derivatives.

Let $(S_t)_{t\geq 0}$ denote the accumulated insurance claim process, $S_t = \sum_{i=1}^{N_t} X_i$. The classical stop-loss contract $(S_T - K)_+$. The payoff of a call option is also $(S_T - K)_+$. Hence, those two covers are identical for an insurance company, willing to transfer risk claims exceeding priority K.

The idea of the pricing model here is to assume that the price of the financial asset has jumps related to the occurrence of catastrophes. This assumption can be validated by stylized facts, e.g. stock price of reinsurance companies and WTC 9/11 in 2001 (see Figure 5, with Munich Re and SCOR - European markets since Wall street has been closed after the catastrophe), oil price and Katrina in August 2005... etc.

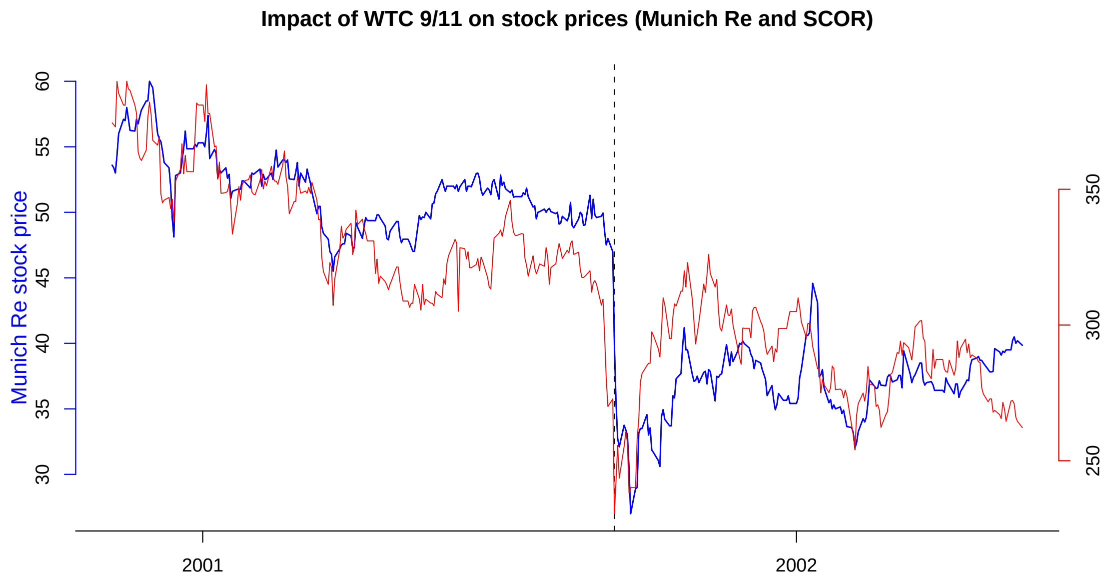

Figure 1: Catastrophe event and stock prices.

The following model and results are based on Quéma *et al.* (2007).

5.1. Description of the model

The occurrence process is a $(\mathcal{F}_t)$-adapted process denoted $(N_t)_{t\geqslant 0}$. Under $\mathbb{P}$, assume that $(N_t)_{t\geqslant 0}$ is an homogeneous Poisson process, with parameter λ, i.e. with stationary and independent increments. Further, recall that $\mathbb{E}_{\mathbb{P}}(N_T) = \lambda T$ and $Var(N_T) = \lambda T$.

At time t that number of catastrophe that had already occurred is N_t. Define the sequence of stopping times $(T_n)_{n\geqslant 0}$ corresponding the dates of occurrence of catastrophes, i.e.

$$T_0 = 0 \text{ and } T_{n+1} = \inf\left\{t \mid t \geqslant T_n, N_t \neq N_{T_n}\right\}.$$

Let $(M_t)_{t\geqslant 0}$ be the compensated Poisson process of $(N_t)_{t\geqslant 0}$, i.e. $M_t = N_t - \lambda t$.

The ith catastrophe has a loss modeled has a positive random variable $\mathcal{F}_{T_i}$-measurable denoted X_i. Variables $(X_i)_{i\geqslant 0}$ are supposed to be integrable, independent and identically distributed. Define $L_t = \sum_{i=1}^{N_t} X_i$ as the loss process, corresponding to the total amount of catastrophes occurred up to time t.

Assume that financial market satisfies the no-arbitrage assumption, and consists in a free risk asset, and a risky asset, with price $(S_t)_{t\geqslant 0}$. Without loss of generality, the value of the risk free asset is assumed to be constant (hence it is chosen as a numeraire). The price of the risky asset is driven by the following diffusion process,

$$dS_t = S_{t^-}\left(\mu dt + \sigma dW_t + \xi dM_t\right) \text{ with } S_0 = 1$$

where $(W_t)_{t\geqslant 0}$ is a Brownian motion under $\mathbb{P}$, independent of the catastrophe occurrence process $(N_t)_{t\geqslant 0}$. Parameters μ and σ^2 are respectively the trend and the volatility of the risky asset, per time unit. Parameter ξ corresponds to the relative variation of the asset value when it jumps.

Note that the stochastic differential equation has the following explicit solution

$$S_t = \exp\left[\left(\mu - \frac{\sigma^2}{2} - \lambda\xi\right)t + \sigma W_t\right](1+\xi)^{N_t}.$$

5.2. Indifference utility

As in Davis (1997) or Schweizer (1997), assume that an investor has a utility function U, and initial endowment ω. The investor is trading both the risky asset and the risk free asset, forming a *dynamic* portfolio $\delta = (\delta_t)_{t\geqslant 0}$ whose value at time t is $\Pi_t = \Pi_0 + \int_0^t \delta_u dS_u. = \Pi_0 + (\delta\cdot S)_t$ where $(\delta\cdot S)$ denotes the stochastic integral of δ with respect to S.

A strategy δ is admissible if there exists $M > 0$ such that $\mathbb{P}\Big(\forall t \in [0,T], (\delta\cdot S)_t \geqslant -M\Big) = 1$, and further if $\mathbb{E}_{\mathbb{P}}\left[\int_0^T \delta_t^2 S_{t^-}^2 dt\right] < +\infty$.

If X is a random payoff, the classical Expected Utility based premium is obtain by solving

$$u(\omega, X) = U(\omega - \pi) = \mathbb{E}_{\mathbb{P}}(U(\omega - X)).$$

Consider an investor selling an option with payoff X at time T,

- either he keeps the option, $u_{\delta^\star}(\omega, 0) = \sup_{\delta\in\mathcal{A}} E_{\mathbb{P}}\Big[U(\omega + (\delta\cdot S)_T)\Big]$,
- either he sells the option, $u_{\delta^\star}(\omega+\pi, X) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}\Big[U(\omega + (\delta\cdot S)_T - X)\Big]$.

The price obtained by indifference utility is the minimum price such that the two quantities are equal, i.e.

$$\pi(\omega, X) = \inf\left\{\pi \in \mathbb{R} \text{ such that } u_{\delta^\star}(\omega+\pi, X) - u_{\delta^\star}(\omega, 0) \geqslant 0\right\}.$$

This price is the minimal amount such that it becomes interesting for the seller to sell the option : under this threshold, the seller has a higher utility keeping the option, and not selling it.

Based on optimal control results, Quéma *et al.* (2007) derived some analytical expression, that can be related to Merton (1976), in the case of exponential utility.

5.3. Following Merton's work

Assume that the asset has no jump, i.e. $dS_t = S_{t^-}(\mu dt + \sigma dW_t)$ (i.e. $\xi = 0$), and that we wish to price a derivative with payoff $\phi(S_T)$, then in the case of exponential utility $u_E(t,\pi) = U_E\left[\pi + \frac{\mu^2 x_0}{2\sigma^2}(T-t)\right]$.

In the case where the asset has jump, i.e. $dS_t = S_{t^-}(\mu dt + \sigma dW_t + \xi dM_t)$ (i.e. $\xi \neq 0$), and that we wish to price a derivative with payoff $\phi(S_T)$, then $u_E(t,\pi) = U_E(\pi + (T-t)C)$ where $C(t)$ satisfies

$$\begin{cases} C(t) = \frac{\alpha x_0}{\xi} + (\alpha - \frac{\sigma^2}{\xi} - \xi\lambda)D - \frac{1}{2x_0}\sigma^2 D^2 \\ 0 = \xi\lambda - \alpha + \frac{\sigma^2}{x_0}D - \xi\lambda \exp\left[-\frac{\xi D}{x_0}\right] \end{cases},$$

with also a border condition, $C(T) = \phi(S_T)$, and where D is related to the optimal strategy, and is obtained also from the previous system.

Here, we wish to price a derivative with payoff $\phi(L_T)$, when the underlying asset has jumps. Then, assuming that the investor has an exponential utility, $U(x) = -\exp(-x/x_0)$,

Theorem 5.1 *Let ϕ denote a $\mathcal{C}^2$ bounded function. If utility is exponential, the value function associated to the primal problem,*

$$u(t,\pi,s,l) = \max_{\delta\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left[U\Big(\Pi_T - \phi(L_T)\Big) \mid \mathcal{F}_t\right]$$

does not depend on s and can be expressed as $u(t,\pi,l) = U\Big(\pi - C(t,l)\Big)$, where C is a function independent of π satisfying

$$\begin{cases} 0 &= \xi\lambda - \mu + \dfrac{\sigma^2 s\delta^\star}{x_0} - \xi\lambda \exp\Big[-\dfrac{\xi s\delta^\star + C(t,l)}{x_0}\Big]\mathbb{E}_{\mathbb{P}}\left(e^{\frac{1}{x_0}C(t,l+X)}\right) \\ \dfrac{\partial C}{\partial t}(t,l) &= \dfrac{\mu x_0}{\xi} + (\mu - \dfrac{\sigma^2}{\xi} - \xi\lambda)s\delta^\star - \dfrac{1}{2x_0}\sigma^2(s\delta^\star)^2 \\ C(T,l) &= \phi(l) \end{cases}$$

where $\delta^\star$ denotes the optimal control.

Proof. Theorem 19 in Quéma *et al.* (2007). ■

5.4. Numerical issues and properties of optimal portfolios

From theorem 5.1 we have to find (C,δ), i.e. 2 functions solutions of an integro-differential equation. Hopefully, using a simple discretization on a finite grid, it is possible to obtain a (stable) numerical approximation of C, and therefore of function C, and thus of the price of the derivative. Note further that ywo nice results have been derived in Quéma *et al.* (2007),

Lemma 5.2 *$C(t,\cdot)$ is increasing if and only if ϕ is increasing.*

Lemma 5.3 *If ϕ is increasing and $\mu > 0$, then the optimal amount of risky asset to be hold when hedging is bounded from below by a strictly positive constant.*

For a numerical example, assume that the trend is null ($\mu = 0$), i.e. amounts hold are uniquely explained by the hedging strategy. Prices are decreasing in x_0, and therefore, increasing with risk aversion (the higher x_0, the lower risk aversion). When $x_0 \to 0$, risk aversion is infinite, and thus, whatever appends, the agent wants to hedge against any losses: the price tends to the super-replication price i.e. $\|\phi\|_\infty$, since if he holds underlying, he might loose money.

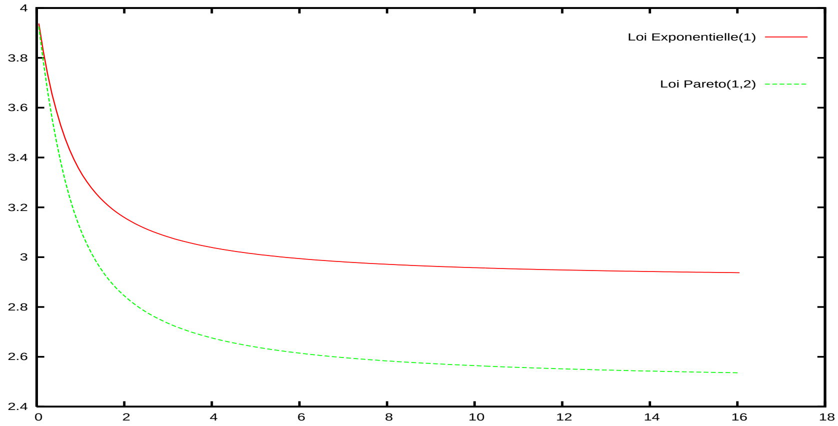

Figure 2: Price as a function of the risk aversion coefficient x_0 with $T = 1$, $\mu = 0$, $\sigma = 0.12$, $\lambda = 4$, $\xi = 0.05$ and $B = 4$

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