Conditional Choquet Capacities on time and on uncertainty

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1. Introduction

Conditioning capacities has been a problem for many years. The pioneer's work of Dempster (1967) and Shafer (1967) who presented the first formula opened the way to many research among which we can cite (forgetting many) Fagin and Halpern (1991) and Jaffray (1992) who introduced another rule (the Full Bayesian Updating Rule), Cohen, Gilboa, Jaffray, Schmeidler (1994) who compared the Bayesian and the Dempster-Shafer rule, etc.

Dempster (1994) used a different approach as he deduced the updating rule (the full Bayesian updating rule or the Bayesian rule depending on the assumptions) from an implicit definition of the conditional Choquet integral. Denneberg followed the usual presentation of the conditional Lebesgue integral (Expectation) in classical measure theory handbooks from which Bayes rule for probabilities obtains. Dempster chose the definition : $E[E^Y(X) - X] = 0$ that is equivalent in the linear case to $E[X - E^Y(X)] = 0$ or to the more usual definition: $E(X) = E[E^Y(X)]$. Each of these formulas yields a different (implicit) definition of the conditional Choquet integral, and, depending on the case, different updating rules (see Kast, Lapied and Toqueboeuf (2007) for a general representation of these results). These defining equations impose some consistency between conditional integrals and non-conditional ones: what is usually called "dynamic consistency" but may take different expressions depending on the model. More generally, all models have imposed some consistency axioms between a priori measures and their updates given information. We know that under the Dynamic Consistency (DC) and the Consequentialism (C) axioms in their general formulation, the valuation will be

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linear (see Sarin and Wakker (1998), Machina (1998), Ghirardato (2002), Lapied and Toquebeuf (2007) and the relevant literature) and degenerate into additive or quasi additive measures after several iterations. In this paper, we obtain a different result and non additive updated capacities, this is because we weaken Consequentialism in the following sense: Counterfactual events are given a zero measure, however, they may interfere with valuation because that they may modify the payoffs ranking. Furthermore, this paper departs from others where, even though dynamic consistency is invoked, it cannot play its full role because the models themselves are not really dynamical. By this we mean that the set of states that represent the future seen from a present date and the information that may arrive "later" are left without any reference to any real timing. In the practice of managing an investment or a project and calculating its ex-ante value, the timing where actions have to be taken when information arrives and may induce options to be exercised, and so on, are fundamental. So that, Time, as well as Uncertainty, is an important part of our description of the future.

Furthermore, time is economically relevant, it is measured by discount factors: whether market ones when a market for bonds exists, or individual ones (preference for present over future consumption). For instance, Koopman (1972) gave seminal conditions for this valuation to be additive (time separability). Then, Gilboa (1989) extended the model to the non-additive case, following the extensions of Subjective Expected Utility models proposed by Schmeidler (1986 and 1989). Notice that Koopman, Gilboa as well as all their followers (notably, Chateauneuf and Rébillé (2004)) exclude uncertainty considerations.

However, in general, future payoffs are contingent on dates in Time, but also on states (events) of Uncertainty. Here again, axioms yielding additive properties to the representation of preferences were extended by Schmeidler (1989) and Gilboa and Schmeidler (1989) among others to the non additive case (capacities). This is why in this paper, we decompose preferences on future payoffs (contingent on futures states in a set Ω) as contingent on two factors: uncertain states and dates. Formally, we write: $\Omega = S \times T$, where S stands for the set of Uncertain states and T represent the set of dates in the future Time under consideration. Preferences of the decision maker could be defined on Ω, but most of the time they are better known on S and on T and it is not clear how to define the global value of a cash flow $X: \Omega = S \times T \rightarrow R$ from the value of uncertain cash payoffs and the value of time contingent time payoffs separately: this is the first step of this paper. The object of the paper is to define

dynamically consistent values and the updating of capacities measuring uncertainty and non additive discount factors measuring time, when information arrives.
In part 2 of the paper, we specify the model : representation of preferences, information, and results from the Ghirardato-Fubini theorem (Ghirardato 1997) on product spaces. In section 3 and 4 we concentrate on the conditioning of capacities on uncertain states and the conditioning of discount factors under a dynamic consistency and a model consistency axioms. We verify that consequentialism is violated even though the updated measures are not additive.

2. The model

We consider that a payoff is a measurable function $X\colon \Omega = S \times T \rightarrow R_+$ where $S = \{s_1, \dots, s_N\}$ represents the set of uncertain states to whom the payoffs are contingent and $T = \{1, \dots, T\}$ the set of uncertain dates, both with the sets of parts, 2^S and 2^T, as σ-algebras. Obviously, a project may have negative payoffs and uncertainty and time may not be perceived as finite sets, we just restrict the problem to this case in order to concentrate on the principles of dynamic valuation, i.e. consistency of preferences with information arrivals.

2.1. Representation of preferences

Given we consider finite spaces, we can deal with simple models of representation of preferences, namely the generalisation (for finite sets) of de Finetti's axioms by Diecidue and Wakker (2002).

Definition 2.1: Two measurable functions X and Y on a set of states E are comonotonic iff and for any two states e and e' : $[X(e) - X(e')][Y(e) - Y(e')] \geq 0$.

Definition 2.2: A comonotonic set of functions is such that all functions in this set are two by two comonotonic (notice that in R^m a comonotonic set is a positive cone generated by m linearly independent comonotonic characteristic functions).

Definition 2.3: A Book is a finite sequence of preferences between two measurable functions (cash flows) (bets or assets, for example): (X_i) , (Y_i) , $i=1\dots n$, such that each X_i is preferred to the corresponding Y_i.

Definition 2.4: A comonotonic Book is formed of functions belonging to the same comonotonic set.

Definition 2.5: A Dutch Book is a Book such that: $\Sigma X_i < \Sigma Y_i$.

We can know state the basis three axioms that define the representations of preferences.

Axiom1: Preferences define a complete preorder on the set on measurable functions.
Axiom 2: There exists a certainty equivalent to any measurable function.
Axiom 3: Preferences allow no comonotonic Dutch Books.

Theorem 2.1 (Diecidue and Wakker, 2002):
For a preference relation on R^m satisfying axioms 1 and 2, such that for all X in R^m their exists a constant equivalent $CE(X) \in R$, the following three statements are equivalent:

(i) *CE(X)* is strictly monotonic, additive on comonotonic vectors and non necessarily additive on non comonotonic vectors.
(ii) There exists a unique capacity such that *CE(X)* is the integral of *X* with respect to this measure.
(iii) *CE(X)* is such that axiom 3 is satisfied.

If axiom 3 is replaced by de Finetti's (1931):
Axiom 3': Preferences allow no Dutch Books.
Then the a special case of theorem 2.1 obtains with *CE* additive and a probability as a special case of a capacity.
The proof of the theorem relies mainly on Diecidue and Wakker's result proving that No Dutch Books (comonotonic or not) imply strict monotonicity of the constant equivalent.

The representation theorem yields the three value functions that we need to represent preferences over the future: Ω. With $X: \Omega = S \times T \rightarrow R_+$ the constant equivalent of X defined by the theorem is:

$V(X) = V(X) = \int_\Omega X \, d\Psi$ if Ψ is a capacity, $\int_\Omega X \, dP$ if P is additive.

$V(X)$ is a present certainty equivalent.

Obviously, Ψ defines two marginal capacities: ν (or μ if it is additive) on R^S and ρ (or π if it is additive) on R^T. From the previous representation theorem, we know that these measures allow to represent preferences of the decision maker over R^S and R^T that satisfy the same axioms as the preferences on R^Ω.

The certainty equivalent of uncertain payoffs is:

$\forall\zeta: R^S \rightarrow R_+,\ E(\zeta) = \int_S \zeta(s)\ d\nu(s)$ if ν is a capacity, $E(\zeta) = \int_S \zeta(s)\ d\mu(s)$ if μ is additive.

The present equivalent of date contingent payoffs is:

$\forall\xi: R^T \rightarrow R_+,\ D(\xi) = \int_T \xi(t)\ d\rho(t)$ if ρ is a capacity, $D(\xi) = \int_T \xi(t)\ d\pi(t)$ if π is additive.

In most economic models these representations are assumed to be known and the problem is to define a representation of preferences overs $R^{S\times T}$ that is consistent with the previous ones. Two obvious candidates are:

$X: \Omega = S\times T \rightarrow R_+,\quad DE(X) = \int_T [\int_S X(s,t)\ d\nu(s)]\ d\rho(t)$ and:

$$ED(X) = \int_S [\int_T X(s,t)\ d\rho(t)\ ]\ d\nu(s).$$

However, it is not the case that Fubini's theorem applies and the two candidates yield the same result $V(X)$, if the measures are not additive. This why, in section 2.3, we shall invoke the Ghirardato-Fubini theorem that will allow us to investigate separately the effect of information arrivals on both E, D and on V.

In the special case where de Finetti'coherence axiom (axiom 3') is satisfied, the cash flows' valuation representing the DM' preferences (constant equivalent) is called the Net Present Value. Indeed, in this case we have:

$$NPV(X) = DE(X) = \int_T [\int_S X(s,t)\ d\mu(s)]\ d\pi(t) = \int_S [\int_T X(s,t)\ d\pi(t)\ ]\ d\mu(s) = ED(X)$$

The equalities are obtained because of Fubini's theorem applied to Lebesgue integrals.

Integrating informational values in the valuation of a cash flow is straightforward: if some information arrives at some date τ, it is valued at that date by the conditional valuation, say E^τ and the original cash flow is replaced (is equivalent) by the cash flow: $(X_1, \ldots, X_{\tau\text{-}1}, E^\tau(X), 0, \ldots, 0))$ then discounted according to the usual conditions in dynamic programming.

The aim of this paper is to extend this result, as far as it can be done, to non-additive valuations.

2.2 Information

Taking future flexibilities and options in an investment or a project into account amounts to integrate the value of the options to modify it in the project's present value. The option is exercised in accordance with information arrivals of the type $[Y=i]$, $i \in I$. Indeed, when information $[Y=i]$ obtains, the DM may modify its preferences over the project's payoffs and hence its valuation. For instance, its aversion to uncertainty (convex capacity) may be reduced, or increased depending on the type of information ("good" or "bad" news). An other example may be that, the DM learning it has more wealth available, its preferences for present consumption may change.

Let us consider a filtration on 2^S: $F = \{F_0, \dots, F_T\}$ with $F_0 = \{\emptyset, S\} \subset F_1 \subset \dots \subset F_T = 2^S$.

Information at date $t = 1, \dots, T$ is given by an F_t-measurable function Y_t that defines a partition I_t of F_t with elements $[Y_t = i_t]$. For the sake of simplifying notations, we'll write: $I_t = (i_t^1, \dots, i_t^{M(t)})$ where for $j = 1, \dots, M(t)$, $i_t^j \in I_t$ and i_t^j stands for the set $[s \in S\ ;\ Y_t(s) \in i_t^j\ ]$.

We assume preferences and conditional preferences satisfy the (Sarin and Wakker (1998)):

Axiom 4 (Model Consistency): Preferences on uncertain payoffs and preferences conditional on information satisfy the same axioms, in our case:

The preferences conditional on information are represented by: $V^i{}_t$, $E^i{}_t$ and $D^i{}_t \equiv D^t$ that are Choquet integrals with respect to capacities: $\Psi^i{}_t$, $\nu^i{}_t$ and $\pi^i{}_t \equiv \pi^t$. (MC)

The conditional Choquet integrals (and the corresponding conditional capacities) have to be defined, at least implicitly, by some Dynamic Consistency requirements.

Such a consistency of valuation with information arrivals can be questioned this way (we drop the time index in the following given we refer here to static models):

If, for some i_t, $V^i{}_t(X) \geq V^i{}_t(X')$ can we have: $V(X) < V(X')$?

The answer is yes, there are cases where we could. For instance, assume the set $[Y=i]$ excludes the set on which $X < X'$, so that on any set in $\sigma([Y=i])$, $X \geq X'$, then, if preferences are monotonic we could have a contradiction between unconditional and conditional valuations[2]. However we need not have one because all the i's are possible and the decision maker may still take into account payoffs for which $X < X'$ and then not prefer X to X'. But if,

[2] For instance it would be the case if the decision maker's preferences satisfy consequentialim.

for all i's, we had $X = X'$ on $[Y=i]^c$, then, consistency with information arrivals will imply that:

$\forall i_t \in I,\ V^i{}_t(X) \geq V^i{}_t(X') \Leftrightarrow V(X) \geq V(X')$.

This equivalence (under the previous condition) is the way Karni and Schmeidler (1983)[3], for instance, expressed Dynamic Consistency (they did it in terms of preferences instead of values as we did and they limited information to a unique value).

We'll require a similar condition, as expressed for example by Nishimura and Osaki (2003):

Axiom 5 (Dynamic Consistency):

$\forall\ \tau = 1, \ldots, T-1$, $\forall\ X, X'$ such that: $X_t(s) = X'_t(s)$, $\forall\ t = 0, \ldots, \tau-1$, $\forall\ s \in S$, $\forall X, X'$,

$$[\forall i_\tau \in I_\tau\ V^i{}_\tau(X) \geq V^i{}_\tau(X')] \Rightarrow V(X) \geq V(X'). \qquad \text{(DC)}$$

Or, in terms of preferences: $[\forall i_\tau \in I_\tau, X \succsim_{i_\tau} X'] \Rightarrow X \succsim X'$.

In order to address the problem of consistently conditioning V, D and E when $V = DE$ or $V = ED$, we invoke the following

2.3 Ghirardato-Fubini theorem

The DM has preferences on R^Ω that are represented by a Choquet integral with respect to a capacity Ψ on 2^Ω:

- $\forall\ X \in R^\Omega\ V(X) = \int_\Omega X\ d\Psi$.

As $\Omega = S \times T$, Ψ yields two marginal capacities, say: ν on 2^S and ρ on 2^T. In turn, these two capacity measures define values of uncertain states-contingent payoffs and of future dates contingent payoffs. These value functions represent preferences over R^S and R^T that satisfy the same axioms than preferences on R^Ω, so they are represented again by Choquet integrals that define:

- $\forall\ t \in T,\ X_t \approx_t E(X_t) = \int_S X_t\, d\nu$.
- $\forall\ s \in S,\ X_s \approx_s D(X_s) = \int_T X_s\, d\rho$.

Mixing up the marginals and the value function representing preferences on R^S and R^T by introducing a hierarchy between the two components that has to be justified, we can define:

[3] But see also : Sarin and Wakker (1998), Machina (1998) and Ghirardato (2002).

- $\forall X \in R^{R\times T}$ $ED(X) = E[D(X)] = \int_S [\int_T X(s,t)\, d\rho(t)]\, d\nu(s)$.

- $\forall X \in R^{R\times T}$ $DE(X) = D[E(X)] = \int_T [\int_S X(s,t)\, d\nu(s)]\, d\rho(t)$.

These value functions define two orders of preferences that have the same properties as the others and can be represented by: $ED(X) = \int_\Omega X\ d\Psi_1$ and $DE(X) = \int_\Omega X\ d\Psi_2$. In general, Ψ_1, Ψ_2 and Ψ will not coincide but in some particular cases that we shall consider.

As we shall see, the hierarchy between preferences on time and on uncertain states can be justified by some hedging properties, then, in these cases, it will be possible to show the coherence of the different preferences and of the measures they define.

Now let us recall some definitions introduced by Ghirardato (1997):

Definition 2.3 (Slice-comonotonicity):

- $X \in R^{S\times T}$ is T-slice (resp. S-slice) comonotonic, if for all t in T, its t-sections on R^S (resp. for all s in S, its s-sections on R^T) are comonotonic.

- $X \in R^{S\times T}$ is slice comonotonic, if all its t-sections and its s-sections are comonotonic.

- A set $F \subset R^{S\times T} = R^\Omega$ is said to be comonotonic if all the s-sections of its characteristic function 1_F are comonotonic, which is equivalent to: all its t-sections are also comonotonic and then to: 1_F is slice comonotonic.

Lemma 2.3: For any T-slice comonotonic X such that $\forall t \in T$, $X_t \in C_k$ where C_k is a comonotonic class, $k \in \{1, \ldots, N!\}$, then ν is represented by a probability measure μ_k and $\forall t \in T$, $E(X_t) = \int_S X_t\, d\nu = \int_S X_t\, d\mu_k$.

For any S-slice comonotonic X such that $\forall s \in S$, $X_s \in C_h$ where C_h is a comonotonic class, $h \in \{1, \ldots, T!\}$, then ρ is represented by a probability measure π_h and $\forall s \in S$, $D(X_s) = \int_T X_s\, d\rho = \int_T X_s\, d\rho_h$.

Proof: If X is T-slice comonotonic, $\forall t \in T$, $X(.,t)$ belongs to some comonotonic class, say C_k, $k=1, \ldots, N!$ of R^S. A comonotonic class C_k is generated by a basis of linearly independent comonotonic characteristic functions of sets $A_k^1 \subset A_k^2 \subset \ldots \subset A_k^N = R^S$. Then, because we assumed all payoffs to be non negative, if X_t is in the comonotonic class C_k :

$$\exists (\alpha^1_t, \ldots, \alpha^N_t) \in R^S_+,\ X_t = \sum_{i=1}^{N} \alpha^i_t\, 1_{A^i_k}.$$

Furthermore, we know that a capacity ν on 2^S is additive on each comonotonic class so that:

$\forall\, C_k,\ k=1,\ \dots\ ,N!\quad \exists\mu_k$ additive so that:

$$\forall\, X_t\in C_k,\ E(X_t) = \int_S X_t\, d\nu = \sum_{i=1}^{N} \alpha^i_t\ \nu(A^i_k) = \int_S X_t\, d\mu_k.$$

Similarly, if X is S-slice comonotonic, then $\forall\, s\in S,\ X(s,.)$ belongs to some comonotonic class, say $C_h,\ h=1,\ \dots\ ,T!$ of R^T that is generated by a basis of linearly independent comonotonic characteristic functions of sets $B_h^1 \subset B_h^2 \subset \dots \subset B_h^T = R^T$ and

$$\exists\, (\beta^1_s,\ \dots,\ \beta^T_s)\in R^T_+,\ X_s = \sum_{i=1}^{T} \beta^i_s\ 1_{B^i_h}.$$

Furthermore, we know that a capacity ρ on T is additive on each comonotonic class so that:

$\forall\ C_h,\ h = 1,\ \dots\ ,T!\quad \exists\pi_h$ additive such that:

$$\forall\ X_s\in C_h,\ D(X_s) = \int_T X_s\, d\rho = \sum_{i=1}^{T} \beta^i_s\ \rho(B^i_h) = \int_T X_s\, d\pi_h.$$

In order to be able to use some of Ghirardato (1997) results, we need to introduce a new axiom on preferences that insure that preferences on R^Ω, R^S and R^T are consistent. Consistency of marginal preferences on state or on date contingent payoffs, and global preferences on cash flows can be expressed by: The measures ν, ρ and Ψ are such that Ψ can be reconstructed from ν and ρ. Obviously, this is requiring too much in general, there are some intertwinements between state and date contingency that may induce some preferences to be modified when future payoffs are seen as whole. However, when no hedging possibilities are available neither on uncertain payoffs nor on certain cash flows, we can impose some consistence of the DM's behaviour (notice that the hedging argument is the one used to justify the comonotonic additivity, or comonotonic independence, or No comonotonic Dutch books axiom).

Axiom 6 (Comonotonic Consistency):

If F is a comonotonic subset of $\Omega = S\times T$, and $(F_t)_{t\in T}$ and $(F_s)_{s\in S}$ are its projections, i.e.: $F_t = \{s\in S \,/\, 1_F(s,\, t) = 1\}$, $F_s = \{t\in T \,/\, 1_F(s,\, t) = 1\}$ then preferences must be such that:

$[\ \forall t\in T\ E(1_F(.,\, t)) = \nu(F_t)$ and $\forall s\in S\ D(1_F(s,\, .) = \rho(F_s)] \Rightarrow V(1_F) = \Psi(F).$

Obviously, the axiom is always satisfied by definition of the marginals if $F = A \times A'$, $A \subset S$, $A' \subset T$. Assume the axiom is not satisfied by a DM, for example, we have for some trajectory s, a measure ρ' on Time $\rho \neq \rho'$ with ρ' convex while ρ is not. This would mean that the DM is more time-variations averse when confronted to a certain time contingent cash flow trajectory s than it is if the payoffs were part of a state and date contingent flow. Obviously, that could be acceptable if F were not comonotonic, but being so, it offers no possibilities for hedging time-variations so that the two different measures are inconsistent.

Proposition 2.3.1: Under the comonotonic consistency axiom, the condition that Ghirardato (1997) dubbed "the Fubini property" is satisfied: For any comonotonic subset F of $2^{S \times T}$,

$\Psi(F) = D(\nu[\{s \in S / (s,t) \in F\}]) = E(\rho[\{t \in T / (s,t) \in F\}])$, or:

$$\int_{S\times T} 1_F(s,t)\, d\psi(s,t) = \int_T d\rho(t) \int_S 1_F(s,t)\, d\nu(s) = \int_S d\nu(s) \int_T 1_F(s,t)\, d\rho(t).$$

Proof: Notice that $\int_S 1_F(s,.)\, d\nu(s)$ is comonotonic with any of the $1_F(.,t)$, $t \in T$, so that there exists an additive probability π_h on T such that:

$$\int_T [\int_S 1_F(s,t)\, d\nu(s)]d\rho(t) = \int_T [\int_S 1_F(s,t)\, d\nu(s)]d\pi_h(t) \text{ and}$$

$$\forall s \in S \int_T 1_F(s,t)\, d\rho(t) = \int_T 1_F(s,t)\, d\pi_h(t).$$

But $\int_T 1_F(.,t)\, d\pi_h(t)$ is comonotonic with any of the $1_F(s,.)$, $s \in S$ so that there exists an additive probability μ_k on S such that :

$$\int_S [\int_T 1_F(s,t)\, d\pi_h(t)]d\nu(s) = \int_S [\int_T 1_F(s,t)\, d\pi_h(t)]d\mu_k(s).$$

Fubini theorem applies and $\int_S [\int_T 1_F(s,t)\, d\pi_h(t)]d\mu_k(s) = \int_T [\int_S 1_F(s,t)\, d\mu_k(s)]d\pi_h(t)$.

This yields the second equality of the lemma.

The first equality obtains because μ_k and π_h define a product probability, say Φ_j on $S\times T$, but Φ_j is an additive representation of Ψ valid on the comonotonic class C_j, $j=1, ..., (N\times T)!$ (in R^{Ω}) to which 1_F belongs. But this is true for any comonotonic class, so the Φ_j's define Ψ. QED

Now, Ghirardato's theorem (in fact, his lemma 3) yields the following decomposition of preferences on $R^{S\times T}$ and preferences on R^S and on R^T:

Proposition 2.3.2 (Ghirardato): Under the comonotonic consistency axiom, if preferences on $R^{S\times T}$ are represented by V (defined by capacity Ψ) and preferences on R^S by E (capacity ν) and on R^T by D (capacity ρ), we have:

1- If X in $R^{S\times T}$ is T-slice comonotonic, then: $V(X) = E[D(X)]$.

Furthermore, for any comonotonic class C_k in R^S containing all the comonotonic t-sections of X, there exists a probability distribution μ_k defining an additive representation E_k of preferences on C_k such that for any X' with all its comonotonic t-sections in C_k:

$V(X') = E_k[D(X')]$.

2- If X in $R^{S\times T}$ is S-slice comonotonic, then: $V(X) = D[E(X)]$.

Furthermore, for any comonotonic class C_h containing all the comonotonic s-sections of X, there exists a probability distribution π_h defining an additive representation D_h of preferences on C_h such that for any X' with all its comonotonic s-sections in C_h:

$V(X') = D_h[E(X')]$.

3- If X in $R^{S\times T}$ is slice comonotonic, then: $V(X) = E[D(X)] = D[E(X)]$.

Interpretations:

The first two results are lemma 3 of Ghirardato (1997). The additive representation (valid only on one comonotonic class) is interpreted this way:

For the first one, consider a model consistent with Gilboa's (1989) idea in which time is measured by an non decreasing, non negative and bounded measure (a capacity in our special case). In this model, uncertainty has not been taken into account of. Now, if we add it at each date, we obtain our model. However, because all the uncertain variables are comonotonic, comonotonic additivity applies and we only need to know the probability distribution that represents it on each comonotonic class. This can, but need be to be, extended to the whole space of uncertain variables, assuming then that de Finetti's coherence axiom applies.

The second formula is the usual discounted expected (here in the sense of a Choquet integral) payoffs. Notice that discount factors (probabilities, here) depend on the comonotonic class in which no hedging of time variations can be obtained.

The last result is the famous Ghirardato-Fubini theorem applied to our model.

In the two next section, we shall address the problem of conditioning NPV expressed in terms of Choquet integrals and derive some results about conditional capacities using the Ghiraradato-Fubini theorem.

3. Conditional valuation of *S*-slice comonotonic cash payoffs

In this section, we consider *S*-slice comonotonic cash payoffs X: $S \times T \rightarrow R_+$, i.e. payoffs such that their time variations along trajectories go all in the same way and hence can't be hedged. From Ghirardato theorem (proposition 2.3.2 , part 2) preferences of the DM are represented for all cash payoffs in a comonotonic class C_h in R^T, by $V = D_h E$ where D_h is linear, expressing the fact that for the cash payoffs at stake the DM is time variations neutral. From now on, we'll drop the *h* index. Notice that, while $E^i{}_t$ and $\nu^j{}_t$ as well as $D^i{}_t \equiv D^t$ and $\pi^j{}_t \equiv \pi^t$ have to be defined, at least implicitly by Dynamic Consistency requirements.

We assume X is a F–measurable process, and in order to introduce the present (as a date 0 that has no other role than defining an eventual present (hence certain) cash amount: $X_0(s_1) = \ldots = X_0(s_N) = X_0$) the set of all possible cash flows generated by X is (we use the same notation X for $X(S \times T)$):

$$X = \begin{pmatrix} X_0(s_1) & \ldots & X_T(s_1) \\ \ldots & & \ldots \\ X_0(s_N) & \ldots & X_T(s_N) \end{pmatrix} \text{ with, } \forall\, t \in T, \forall\, s \in S, X_t(s) \geq 0 \text{ and } X_0(s_1) = \ldots = X_0(s_N) = X_0.$$

$$V(X) = \sum_{t \in T} [\sum_{s \in S} X_t(s) \Delta \nu(s)] \pi(t) = \sum_{t \in T} E_t(X) \pi(t),$$

where: $\forall\, t \in T,\ E_t(X) = \sum_{s \in S} X_t(s) \Delta \nu(s)$, with the usual notation for a Choquet integral:

if, for instance, $X(s_1) \leq \ldots \leq X(s_N)$, then: $\Delta \nu(s_n) = \nu(s_n \cup \ldots \cup s_N) - \nu(s_{n+1} \cup \ldots \cup s_N)$, $s_{N+1} = \varnothing$.

Define:

$$EC(X) = \begin{pmatrix} E_0(X) & \ldots & E_T(X) \\ \ldots & & \ldots \\ E_0(X) & \ldots & E_T(X) \end{pmatrix},$$

with $\pi(0) = 1$,we have:

$V[EC(X)] = V(X)$.

Therefore, $EC(X)$ is a certainty equivalent of X.

From the assumption of Model Consistency we have the same type of value functions for a given information at some time τ.

$\forall\ \tau \in T, \forall\ i_\tau \in I_\tau,\ V^{i_\tau}(X) = \sum_{t \in T} [\sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s)] \pi^\tau(t) = \sum_{t \in T} E_t^{i_\tau}(X) \pi^\tau(t),$

where: $\forall\ t = 0,\ldots,\tau - 1,\ \pi^\tau(\mathrm{t}) = 0,\ \pi^\tau(\tau) = 1,\ \forall\ t = \tau,\ldots,T,\ E_t^{i_\tau}(X) = \sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s),$

and define:

$$EC^{i_\tau}(X) = \begin{pmatrix} E_\tau^{i_\tau}(X) & \ldots & E_T^{i_\tau}(X) \\ \ldots & & \ldots \\ E_\tau^{i_\tau}(X) & \ldots & E_T^{i_\tau}(X) \end{pmatrix},$$

with:

$V^{i_\tau}[EC^{i_\tau}(X)] = V^{i_\tau}(X).$

3.1. Dynamic consistency

In section 2, we have introduced a weak definition of Dynamic Consistency (Axiom 5), which gave a link between unconditional and conditional valuations.

Proposition 3.1.1: (DC) implies: $\forall\ \tau = 1,\ldots,T-1,\ \forall\ t = \tau,\ldots,T,$

$$\sum_{i_\tau \in I_\tau} [\sum_{s \in S} X_t(s) \Delta \nu^{i_\tau}(s)] \Delta \nu(i_\tau) = \sum_{s \in S} X_t(s) \Delta \nu(s) \qquad (3.1)$$

Proof: $\forall\ \tau = 1,\ldots,T-1$, define Z^τ by the following:

$\forall\ t = 0,\ldots,\tau-1,\ \forall\ s \in S,\ X_t(s) = Z^\tau_t(s)$, and

$\forall\ t = \tau,\ldots,T,\ \forall\ s \in S,\ \exists\ l \in \{1, \ldots, M(\tau)\}$, such that $s \in i_\tau^l$ and then $Z_t^\tau(s) = E_t^{i_\tau^l}(X)$.

$\forall\ i_\tau \in I_\tau,\ V^{i_\tau}(Z^\tau) = \sum_{t=\tau}^{T} [\sum_{i \in I_\tau} E_t^i(X) \Delta \nu^{i_\tau}(i)] \pi^\tau(t).$

Suppose w.l.o.g. that, for a date $t = \tau,\ldots,T$:

$E_t^{i_\tau^1}(X) \leq \ldots \leq E_t^{i_\tau^{m(\tau)}}(X) \leq \ldots \leq E_t^{i_\tau^{M(\tau)}}(X)$, for which $i_\tau = i_\tau^{m(\tau)}$.

Then:

$$\sum_{i \in I_\tau} E_t^i(X) \Delta \nu^{i_\tau}(i) = \sum_{l=1}^{M(\tau)} E_t^{i_\tau^l}(X) [\nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^l) - \nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^{l-1})]$$

$$= \sum_{l=1}^{m(\tau)-1} E_t^{i_\tau^l}(X)[\nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^l) - \nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^{l-1})]$$

$$+E_t^{i_\tau^{m(\tau)}}(X)[\nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^{m(\tau)}) - \nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^{m(\tau)-1})]$$

$$+ \sum_{l=m(\tau)+1}^{M(\tau)} E_t^{i_\tau^l}(X)[\nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^l) - \nu^{i_\tau^{m(\tau)}}(i_\tau^1 \cup \ldots \cup i_\tau^{l-1})].$$

With the normalisation of conditional capacities:

$i \subset A \Rightarrow \nu^i(A) = 1$, $i \cap A = \varnothing \Rightarrow \nu^i(A) = 0$,

$$\sum_{i \in I_\tau} E_t^i(X)\Delta\nu^{i_\tau}(i) = E_t^{i_\tau^{m(\tau)}}(X) = E_t^{i_\tau}(X).$$

It follows that:

$$\forall\, i_\tau \in I_\tau,\ V^{i_\tau}(Z^\tau) = \sum_{t=\tau}^{T} E_t^{i_\tau}(X)\pi^\tau(t) = V^{i_\tau}(X).$$

Therefore, with (DC), we have: $V(Z^\tau) = V(X)$, which implies:

$$\sum_{t=0}^{\tau-1}[\sum_{s\in S} X_t(s)\Delta\nu(s)]\pi(t) + \sum_{t=\tau}^{T}[\sum_{i_\tau \in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau)]\pi(t) = \sum_{t\in T}[\sum_{s\in S} X_t(s)\Delta\nu(s)]\pi(t),$$

and then:

$$\sum_{t=\tau}^{T}[\sum_{i_\tau \in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau)]\pi(t) = \sum_{t=\tau}^{T}[\sum_{s\in S} X_t(s)\Delta\nu(s)]\pi(t).$$

This equality is satisfied for any X, and then it should be true for each date t:

$$\sum_{i_\tau \in I_\tau} E_t^{i_\tau}(X)\Delta\nu(i_\tau) = \sum_{i_\tau \in I_\tau}[\sum_{s\in S} X_t(s)\Delta\nu^{i_\tau}(s)]\Delta\nu(i_\tau) = \sum_{s\in S} X_t(s)\Delta\nu(s) \qquad (3.1)$$

QED

3.2 Updating capacities

Relation (3.1) is a condition on the DM's (representation) of preferences that yields an implicit definition of conditional Choquet expectation.

Proposition 3.2.1: Under relation (3.1), for any $i \in I_\tau$, the conditional capacity of a set $A \in F_t$, $t > \tau$, is given by:

(i) If $A \subset i$, $\nu^i(A) = \dfrac{\nu(A \cap i)}{\nu(i)}$ (Bayes updating rule).

(ii) If $A^C \subset i$, $\nu^i(A) = \dfrac{\nu(A \cup i^C) - \nu(i^C)}{1 - \nu(i^C)}$ (Dempster-Schafer updating rule).

Proof: With these notations, relation (3.1), became:

$$\sum_{i \in I_\tau} [\sum_{s \in S} X_t(s) \Delta \nu^i(s)] \Delta \nu(i) = \sum_{s \in S} X_t(s) \Delta \nu(s)$$

For the determination of conditional capacities, payoffs shall be defined as characteristic functions. For $A \in F_t$, the payoffs of X_t at date t are 1_A, and then:

(3.2) $\nu(A) = \sum_{i=i^1}^{i^{M(\tau)}} \nu^i(A) \Delta \nu(i)$.

The conditional capacity $\nu^i(A)$ can only be calculated in two situations:

(i) When $A \subset i$, the "comonotonic" case (because 1_A and 1_i are comonotonic uncertain variables). In this case, $\nu^i(A) \geq 0$ and $\nu^j(A) = 0$, for $j \in I_\tau$, $j \neq i$. Relation (3.2) implies: $\nu(A) = \nu^i(A)\ \nu(i)$, and then $\nu^i(A) = \dfrac{\nu(A)}{\nu(i)} = \dfrac{\nu(A \cap i)}{\nu(i)}$, Bayes formula.

(ii) When $A^C \subset i$, the "antimonotonic" case (because 1_A and 1_i are anticomonotonic, i.e. 1_A and -1_i are comonotonic uncertain variables). In this case, $\nu^i(A) \leq 1$ and $\nu^j(A) = 1$, for $j \in I_\tau, j \neq i$. Relation (3.2) implies:

$\nu(A) = \nu^i(A) + [1 - \nu^i(A)]\ \nu(i^C)$, and then $\nu^i(A) = \dfrac{\nu(A) - \nu(i^C)}{1 - \nu(i^C)} = \dfrac{\nu(A \cup i^C) - \nu(i^C)}{1 - \nu(i^C)}$,

Dempster-Shafer formula. QED

3.3. Consequentialism

An other familiar consistency condition known as consequentialism (Hammond (1989)) is usually imposed on preferences. It is well known, however (see, for instance Sarin and Wakker (1998), Machina (1998), Karni and Schmeidler (1991), Ghirardato (2002), Lapied and Toquebeuf (2007)) that Model consistency, Dynamic consistency and Consequentialism imply additive (or quasi always additive) models. Ours is not, under the two first assumptions, then it must be that Consequentialism is not satisfied, as we show below.

Definition 3.3.1 (Consequentialism in a dynamic setting):

$\forall\ \tau = 0,\ldots,T$, $\forall i_\tau \in I_\tau$, $[\forall\ t = \tau,\ldots,T, \forall\ s \in i_\tau, X_t(s) = X'_t(s)] \Rightarrow X \approx_{i_\tau} X'$ (C)

Proposition 3.3.1: $V = DE$ does not satisfy (C).

Proof: Let us consider $S = \{s_1,s_2,s_3,s_4\}$, $I_1 = \{i^1,i^2\}$, $i^1 = \{s_1,s_2\}$, $i^2 = \{s_3,s_4\}$, $T = \{0,1,2\}$ and two risks X, X', with the following payoffs:

$X_0 = X'_0 = 12$,

$X_1(\{s_1\}) = X_1(\{s_2\}) = X'_1(\{s_1\}) = X'_1(\{s_2\}) = 10$,

$X_1(\{s_3\}) = X_1(\{s_4\}) = X'_1(\{s_3\}) = X'_1(\{s_4\}) = 9$,

$X_2(\{s_1\}) = 8$, $X_2(\{s_2\}) = 4$, $X'_2(\{s_1\}) = 0.2$, $X'_2(\{s_2\}) = 0.4$,

$X_2(\{s_3\}) = X'_2(\{s_3\}) = 2$, $X_2(\{s_4\}) = X'_2(\{s_4\}) = 1$.

Because X and X' are S-slice comonotonic cash payoffs, we can apply DE valuation.

For these risks, consequentialism, implies that $V_1^{i^2}(X) = V_1^{i^2}(X')$.

We have:

$$V_1^{i^2}(X) = 9 + \pi^1(2)\{1 \times [1 - \nu^{i^2}(\{s_1,s_2,s_3\})] + 2 \times \nu^{i^2}(\{s_1,s_2,s_3\})\},$$

$$V_1^{i^2}(X') = 9 + \pi^1(2)\{1 \times [1 - \nu^{i^2}(\{s_3\})] + 2 \times \nu^{i^2}(\{s_3\})\}.$$

From Proposition 3.2.1:

- Because $\{s_3\} \subset i^2$ its conditional capacity is given by Bayes updating rule:

$$\nu^{i^2}(\{s_3\}) = \frac{\nu(\{s_3\})}{\nu(i^2)}.$$

- Because $\{s_1,s_2,s_3\}^C = \{s_4\} \subset i^2$ its conditional capacity is given by Dempster-Schafer updating rule:

$$\nu^{i^2}(\{s_1,s_2,s_3\}) = \frac{\nu(\{s_1,s_2,s_3\}) - \nu(i^1)}{1 - \nu(i^1)}.$$

Let $\pi^1(2) = 0.9$, and ν be a convex (non linear) capacity with:

$\nu(\{s_3\}) = 0.3$, $\nu(i^1) = 0.5$, $\nu(i^2) = 0.4$, $\nu(\{s_1,s_2,s_3\}) = 0.9$.

We obtain: $V_1^{i^2}(X) = 10.62 > V_1^{i^2}(X') = 10.575$,

which is a contradiction to Consequentialism. QED

Non consequentialism open the way to non linearity of the valuation.

Proposition 3.3.2: $V = DE$ does not reduce to discounted expected cash flows.

Proof: We consider the same example as in proposition 3.3.1.

We have:

$$V_1^{i^1}(X) = 10 + \pi^1(2)\{4 \times [1 - \nu^{i^1}(\{s_1\})] + 8 \times \nu^{i^1}(\{s_1\})\},$$

$$V_1^{i^1}(X') = 10 + \pi^1(2)\{0.2 \times [1 - \nu^{i^1}(\{s_2, s_3, s_4\})] + 0.4 \times \nu^{i^1}(\{s_2, s_3, s_4\})\}.$$

From Proposition 3.2.1:

- Because $\{s_1\} \subset i^1$ its conditional capacity is given by Bayes updating rule:

$$\nu^{i^1}(\{s_1\}) = \frac{\nu(\{s_1\})}{\nu(i^1)}.$$

- Because $\{s_2, s_3, s_4\}^C = \{s_1\} \subset i^1$ its conditional capacity is given by Dempster-Schafer updating rule:

$$\nu^{i^1}(\{s_2, s_3, s_4\}) = \frac{\nu(\{s_2, s_3, s_4\}) - \nu(i^2)}{1 - \nu(i^2)}.$$

If we add the following values to the definition of ν:

$\nu(\{s_1\}) = 0.3$, $\nu(\{s_2, s_3, s_4\}) = 0.6$, we obtain:

$\sum_{s \in S} X_2(s) \Delta \nu^{i^1}(s) = 6.4$, $\sum_{s \in S} X_2(s) \Delta \nu^{i^2}(s) = 1.8$, and

$\sum_{s \in S} X'_2(s) \Delta \nu^{i^1}(s) = \frac{0.8}{3}$, $\sum_{s \in S} X'_2(s) \Delta \nu^{i^2}(s) = 1.75$.

Relation (3.1) is trivially satisfied for $\tau = t = 1$, we have only to consider the case where $\tau = 1$ and $t = 2$:

$$\sum_{i \in I_1} [\sum_{s \in S} X_2(s) \Delta \nu^i(s)] \Delta \nu(i) = 4.1 = \sum_{s \in S} X_2(s) \Delta \nu(s),$$

$$\sum_{i \in I_1} [\sum_{s \in S} X'_2(s) \Delta \nu^i(s)] \Delta \nu(i) = 0.86 = \sum_{s \in S} X'_2(s) \Delta \nu(s).$$

Therefore, relation (3.1) is consistent with the (non linear) capacity ν and with the conditional capacities defined by proposition 3.2.1. QED

4. Conditional valuation of *T*-slice comonotonic cash payoffs

In this section we concentrate on cash payoffs with all their *t*-sections in the same comonotonic class in R^S, say C_k, hence their uncertain variations can't be hedged. As a consequence of Ghirardato's theorem (proposition 2.3.2, part 1), preferences on $R^{S \times T}$ are represented by a $V = E_k D$ valuation function, with E_k a Lebesgue integral with respect to a probability distribution μ_k on 2^S and D a Choquet integral with respect to a capacity ρ. In the following, we drop the index k.

We know that information only bears on 2^S, its only influence on preferences on R^T is on the date at which it is obtained. Otherwise stated: $D^i{}_t \equiv D^t$ is a Choquet integral with respect to a capacity ρ^t that is contingent on date t only, while $E^i{}_t$ is the usual conditional expectation and $\mu^i{}_t$ is obtained by the probabilistic Bayes'rule.

We have first to adapt the valuation to this new decomposition between time and uncertainty:

$$V(X) = \sum_{s\in S}[\sum_{t\in T} X_t(s)\Delta\rho(t)]\mu(s) = \sum_{s\in S} D_s(X)\mu(s),$$

where: $\forall\, s \in S,\ D_s(X) = \sum_{t\in T} X_t(s)\Delta\rho(t)$.

Define:

$$ET(X) = \begin{pmatrix} D_{s_1}(X) & 0 & \dots & 0 \\ \dots & \dots & & \dots \\ D_{s_N}(X) & 0 & \dots & 0 \end{pmatrix},$$

with $\rho(0) = 1$, we have:

$V[ET(X)] = V(X)$.

Therefore, $ET(X)$ is a temporal equivalent of X.

By the assumption of Model Consistency, we have the same type of value functions for a given pair time-information.

$$\forall\ \tau\in T, \forall\ i_\tau\in I_\tau,\ V^{i_\tau}(X) = \sum_{s\in S}[\sum_{t\in T} X_t(s)\Delta\rho^\tau(t)]\mu^{i_\tau}(s) = \sum_{s\in S} D_s^\tau(X)\mu^{i_\tau}(s),$$

where: $\forall\ s \in i_\tau,\ D_s^\tau(X) = \sum_{t\in T} X_t(s)\Delta\rho^\tau(t)$,

and define:

$$ET^\tau(X) = \begin{pmatrix} D_{s_1}^\tau(X) & 0 & \dots & 0 \\ \dots & \dots & & \dots \\ D_{s_N}^\tau(X) & 0 & \dots & 0 \end{pmatrix},$$

with $\rho^\tau(\tau) = 1$, we have: $V^{i_\tau}[ET^\tau(X)] = V^{i_\tau}(X)$.

We want to give a null ponderation to the payoffs in the past, when we discount. For this, suppose that: $\forall\ F \subset T,\ \rho^\tau(F) = \rho^\tau(F \cap \tau^+)$, where $\tau^+ = \{\tau,\dots,T\}$.

4.1 Dynamic consistency

Dynamic consistency has the same definition than in section 2.

Proposition 4.1.1: (DC) implies :∀ τ= 1,…,T–1, ∀ $s \in S$,

$$\sum_{t\in \tau^-} X_t^\tau(s)\Delta\rho(t) = \sum_{t\in T} X_t(s)\Delta\rho(t) \qquad (4.1)$$

where: $X_t^\tau(s) = X_t(s),\ if\ \ t\in \tau^- - \{\tau\},\ X_t^\tau(s) = D_s^\tau(X),\ if\ \ t=\tau,$

τ^- = {0,…,τ}, and

∀ $s \in S$, $D_s^\tau(X) = \sum_{t\in T} X_t(s)\Delta\rho^\tau(t)$.

Proof: ∀ τ= 1,…,T–1, define:

$$Z^\tau = \begin{pmatrix} X_0 & \dots & X_{\tau-1}(s_1) & D_{s_1}^\tau(X) & 0 & \dots & 0 \\ \dots & & \dots & & \dots & & \dots \\ X_0 & \dots & X_{\tau-1}(s_N) & D_{s_N}^\tau(X) & 0 & \dots & 0 \end{pmatrix}.$$

∀ τ= 1,…,T–1, ∀ t = 0,…,τ–1, ∀ $s \in S$, $X_t(s) = Z^\tau_t(s)$.

For any $s \in S$, consider a permutation of the t = 0,…,τ – 1 such that:

$$0 \le X_{(0)}(s) \le \ldots \le X_{(k)}(s) \le D_s^\tau(X) \le X_{(k+1)}(s) \le \ldots \le X_{(\tau-1)}(s).$$

$$\sum_{t\in T} Z_t^\tau \Delta\rho^\tau(t) = \sum_{(t)=(0)}^{(k)} X_{(t)}(s)\{\rho^\tau[(t),\ldots,(\tau-1),\tau] - \rho^\tau[(t+1),\ldots,(\tau-1),\tau]\}$$

$$+ D_s^\tau(X)\{\rho^\tau[(k+1),\ldots,(\tau-1),\tau] - \rho^\tau[(k+1),\ldots,(\tau-1)]\}$$

$$+ \sum_{(t)=(k+1)}^{(\tau-1)} X_{(t)}(s)\{\rho^\tau[(t),\ldots,(\tau-1)] - \rho^\tau[(t+1),\ldots,(\tau-1)]\}$$

$$= D_s^\tau(X)\rho^\tau(\tau) = D_s^\tau(X).$$

∀ $i_\tau \in I_\tau$, $V^{i_\tau}(Z^\tau) = \sum_{s\in S}[\sum_{t\in T} Z_t^\tau \Delta\rho^\tau(t)]\mu^{i_\tau}(s) = \sum_{s\in S} D_s^\tau(X)\mu^{i_\tau}(s) = V^{i_\tau}(X)$.

Therefore, with (DC), we have: $V(Z^\tau) = V(X)$, which implies:

$$\sum_{s\in S}[\sum_{t\in \tau^-} X_t^\tau(s)\Delta\rho(t)]\mu(s) = \sum_{s\in S}[\sum_{t\in T} X_t(s)\Delta\rho(t)]\mu(s),$$

where: $X_t^\tau(s) = X_t(s),\ if\ \ t\in \tau^- - \{\tau\},\ X_t^\tau(s) = D_s^\tau(X),\ if\ \ t=\tau$.

This equality is satisfied for any X, and then it should be true for each state s:

∀ τ= 1,…,T–1, ∀ $s \in S$,

$$\sum_{t\in\tau^-} X_t^\tau(s)\Delta\rho(t) = \sum_{t\in T} X_t(s)\Delta\rho(t) \qquad (4.1)$$

QED

4.2 "Upstating" capacities on Time

If updating means that we modify the measure of uncertainty according to information at some date, then we dubb "upstating" the fact that we modify the measure of time according to information (on the set of states) at the date at which it is obtained. We still assume that the DM's preferences satisfy the axiom of Dynamic Consistency, i.e. relation (4.1) so that we have:

Proposition 4.2.1: Under relation (4.1), for $F \subset T$, with $\tau^- = \{0,\ldots,\tau\}$, and $\tau^+ = \{\tau,\ldots,T\}$, the "upstated" discount factors are given by:

(i) If $\rho(F) \geq \rho[(F\cap\tau^-)\cup\{\tau\}]$:

$$\rho^\tau(F\cap\tau^+) = \frac{\rho(F) - \rho[(F\cap\tau^-)\cup\{\tau\}] + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

(ii) If $\rho(F) \leq \rho[(F\cap\tau^-)\cup\{\tau\}]$:

$$\rho^\tau(F\cap\tau^+) = \frac{\rho(F) - \rho(F\cap\tau^-)}{\rho[(F\cap\tau^-)\cup\{\tau\}] - \rho(F\cap\tau^-)}.$$

Proof: We drop the reference to the state s in relation (4.1) wlog:

(4.2) $$\sum_{t\in\tau^-} X_t^\tau\Delta\rho(t) = \sum_{t\in T} X_t\Delta\rho(t).$$

For $F \subset T$ and $X = 1_F$, we have:

$\sum_{t\in T} X_t\Delta\rho(t) = \rho(F)$, and $D^\tau(X) = \sum_{t\in\tau^+} X_t\Delta\rho^\tau(t) = \rho^\tau(F\cap\tau^+)$.

We have to consider two cases:

(i) $\rho^\tau(F\cap\tau^+) \geq 1$, then:

$$\sum_{t\in\tau^-} X_\tau^t\Delta\rho(t) = \rho[(F\cap\tau^-)\cup\{\tau\}] + \rho(\{\tau\})[\rho^\tau(F\cap\tau^+) - 1] = \rho(F),$$

(ii) $\rho^\tau(F\cap\tau^+) \leq 1$, then:

$$\sum_{t\in\tau^-} X_\tau^t\Delta\rho(t) = \rho[(F\cap\tau^-)\cup\{\tau\}]\rho^\tau(F\cap\tau^+) + \rho(F\cap\tau^-)[1-\rho^\tau(F\cap\tau^+)] = \rho(F).$$

These relations yield the "upstating" formulas under the equivalent conditions given in the proposition. QED

In the more familiar case where $F = \{0,\ldots,T\}$ we have the following:

Corollary 4.2.1: Under relation (4.1), for $F = \{0,\ldots,T\}$, the "upstated" discount factors are given by:

$$\rho^{\tau}(\{\tau,\ldots,T\}) = \frac{\rho(\{0,\ldots,T\}) - \rho(\{0,\ldots,\tau\}) + \rho(\{\tau\})}{\rho(\{\tau\})}.$$

The interpretations of these "upstating" formula are not straightforward. An important point to note, is that in contrast with the additive case, i.e. the usual compound discount factors formula, which, in the corollary, would yield: $\pi^{\tau}(\{\tau,\ldots,T\}) = \dfrac{\pi(\{\tau,\ldots,T\})}{\pi(\{\tau\})}$, the value of the past does count. We shall come back to this in section 4.3 where we'll see that it is what violates consequentialism. Here, in contrast with the case of conditioning uncertainty, comonotonicity doesn't plays a direct role. In fact it does through the ranking of values obtained before and after information is revealed at date τ.

4.3. Consequentialism

With the definition of consequentialism in section 3, we have:

Proposition 4.3.1: $V = ED$ does not satisfy (C).

Proof: Let us consider the following certain payoffs X and X':

$X_0 = 1, X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1,$ $\quad$ $X'_0 = 0, X'_1 = 1, X'_2 = 0, X'_3 = 1, X'_4 = 1,$

or: $\quad X = 1_F, F = \{0,3,4\},$ $\quad$ $X' = 1_H, H = \{1,3,4\}.$

Let ρ be a capacity such that: $\rho(F) = \rho(0,3,4) > \rho(0,2) = \rho[(F \cap \tau^-) \cup \{\tau\}]$, and

$$\rho(H) = \rho(1,3,4) < \rho(1,2) = \rho[(F \cap \tau^-) \cup \{\tau\}].$$

From Proposition 4.2.1,

$$\rho^2(F \cap \tau^+) = \rho^2(3,4) = \frac{\rho(0,3,4) - \rho(0,2) + \rho(2)}{\rho(2)} > 1 \qquad \text{case (i)}$$

$$\rho^2(H\cap\tau^+)=\rho^2(3,4)=\frac{\rho(1,3,4)-\rho(1)}{\rho(1,2)-\rho(1)}<1 \qquad \text{case (ii)}$$

For these payoffs, consequentialism, implies that $V_2(X)=V_2(X')$. We have:

$$V_2(X)=\sum_{t=0}^{4}X_t\Delta\rho^2(t)=D^2(X)=\rho^2(F)$$

$$=\rho^2(F\cap\tau^+)=\rho^2(3,4)=\frac{\rho(0,3,4)-\rho(0,2)+\rho(2)}{\rho(2)}>1,$$

$$V_2(X')=\sum_{t=0}^{4}X'_t\Delta\rho^2(t)=D^2(X')=\rho^2(H)$$

$$=\rho^2(H\cap\tau^+)=\rho^2(3,4)=\frac{\rho(1,3,4)-\rho(1)}{\rho(1,2)-\rho(1)}<1,$$

We obtain: $V_2(X)>V_2(X')$, which is a contradiction to consequentialism. QED

Proposition 4.3.2: $V = ED$ does not reduce to linear discounted cash flows.

Proof: We consider the same example as in proposition 4.3.1.

We have:

$$\sum_{t=0}^{2}X_t^2\Delta\rho(t)=1\times[\rho(0,2)-\rho(2)]+D^2(X)\times\rho(2)=\rho(0,3,4)=\sum_{t=0}^{4}X_t\Delta\rho(t),$$

$$\sum_{t=0}^{2}X'^2_t\Delta\rho(t)=D^2(X')\times[\rho(1,2)-\rho(1)]+1\times\rho(1)=\rho(1,3,4)=\sum_{t=0}^{4}X'_t\Delta\rho(t),$$

therefore relation (4.1) is satisfied for $\tau = 2$.

The same results holds for $\tau = 1$ and $\tau = 3$.

Relation (4.1) is then consistent with a (non linear) capacity ρ and with the conditional capacities defined by proposition 4.2.1. QED

5. Conclusions

The conclusion are left to the reader…