Risk apportionment and Stochastic dominance

by

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Joint work with
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Plan of the talk.

I Risk aversion, prudence, temperance.

I1. The traditional approach and source of its inconsistencies.

I2 A neglected paper: Gross-Menezes-Trusler in AER 1980.

I3 The notion of "risk apportionment" (pain).

I4 The translation in utility terms.

II The extension to higher order derivatives

III Risk apportionment and S.D.

IV. Extensions.

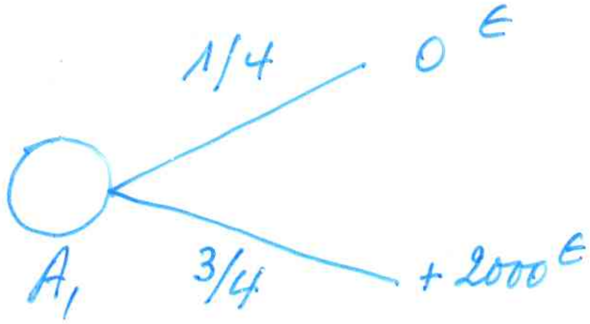
IV 1. The multivariate utility function

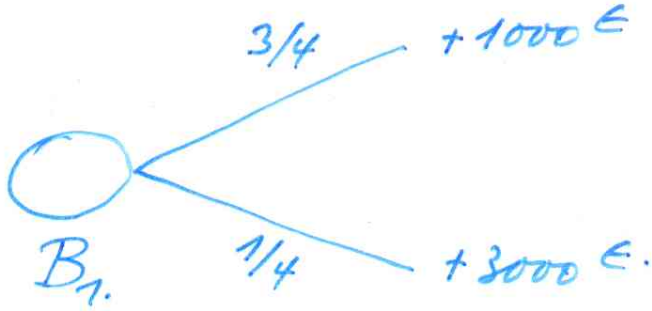
IV 2. Correlation aversion.

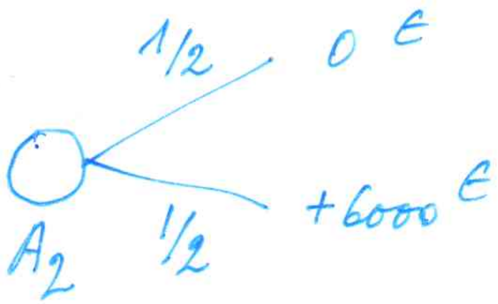
IV 3 Non E-U models.

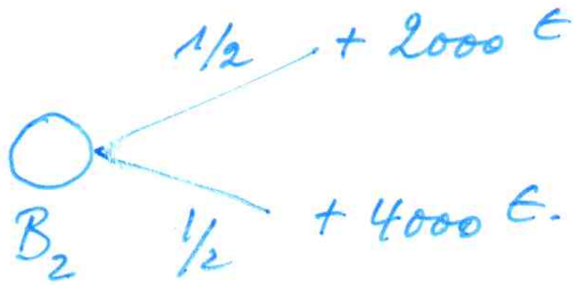
IV 4. The intensity of the effects.

IV 5 Experimental Economics.









$N = 45$

B_1 : 33

B_2 : 35.

$B_1 B_2$: 30

$A_1 B_2$: 3

$B_1 A_2$: 2.

Ind 1 B_2 : 2.

I Risk aversion, prudence, temperance

I 1. The traditional approach

I.1.1. Risk aversion.

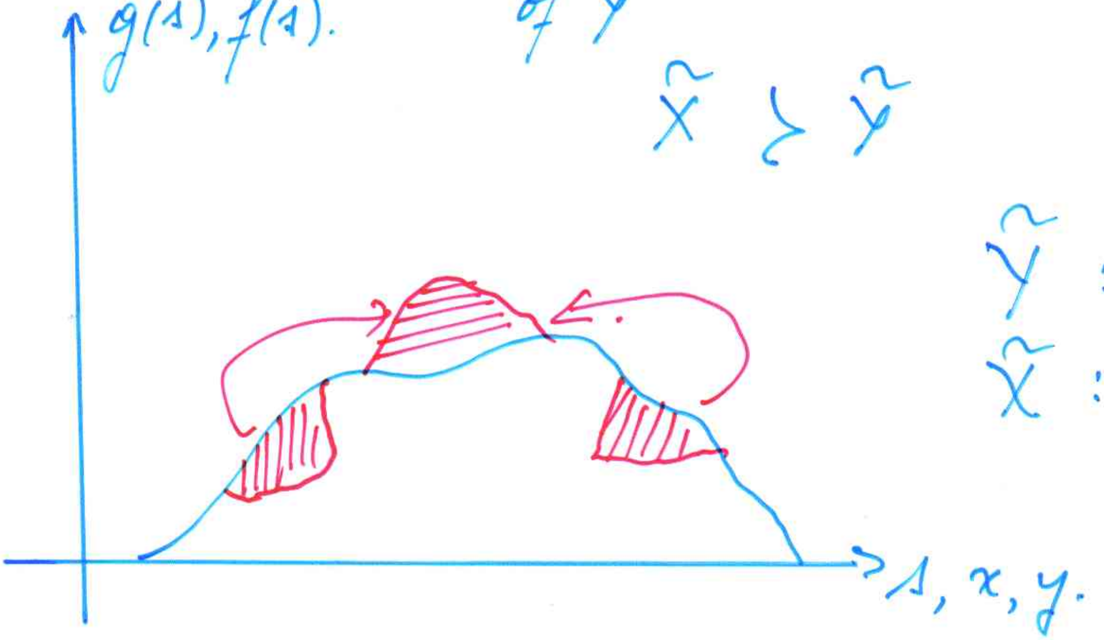
a) The most used definition is in the E-U model: $U'' < 0$.

b) Yet there exists a "model-free" approach (Rothschild-Stiglitz).

- weak form: $E(\tilde{x}) \succsim \tilde{x}$
- strong form

if $\tilde{X}$ is a Mean Preserving Contraction of $\tilde{Y}$

$$\tilde{X} \succsim \tilde{Y}$$



$\tilde{Y} : g(s)$
$\tilde{X} : f(s)$.

In the E-U model $\tilde{X} \succsim \tilde{Y} \Leftrightarrow U'' < 0$.

The definition is preference based.
Then: implication for choices.

I.1.2 Prudence

Basic reference: Kimball, Econometrica, 1990.

Predecessors: Leland (1968) Sandmo (1970)
Drèze-Modigliani (1972)

Summary of the idea: compare 2 DECISION problems in the E-U. framework

•) $\max_{c_1} \; u(c_1) + u((y_1 - c_1) + y_2)$

$$c_1^* = \frac{y_1 + y_2}{2} = c_2^*$$

•) $\max_{c_1} \; u(c_1) + E\left[u((y_1 - c_1) + \tilde{y}_2)\right]$

$$E(\tilde{y}_2) = y_2$$

Result: $\hat{c}_1 < c_1^* \iff u''' > 0$

Hence the property $u''' > 0$ is defined from a decision problem.

I 1.3 Temperance

Basic reference : Kimball, New Palgrave Dict. of Money and Finance (1992)

"The precautionary motive for holding assets"

"An unavoidable risk lead(s) an agent to reduce exposure to another risk even if the 2 risks are statistically independent"

Compare 2 Decision problems.

$$\underset{\alpha}{Max} \; E\left[u(w+\alpha\tilde{\varepsilon})\right] \qquad E(\tilde{\varepsilon})>0$$

$$\alpha^*$$

$$\underset{\alpha}{Max} \; E\left[u(w+\tilde{\theta}+\alpha\tilde{\varepsilon})\right] \qquad E(\tilde{\theta})=0.$$

$$\alpha^{**} \qquad \tilde{\theta} \perp \tilde{\varepsilon}$$

$\alpha^{**} < \alpha^* =$ temperance.

$u^{(4)} < 0$ is necessary.

Summary: the "inconsistency"

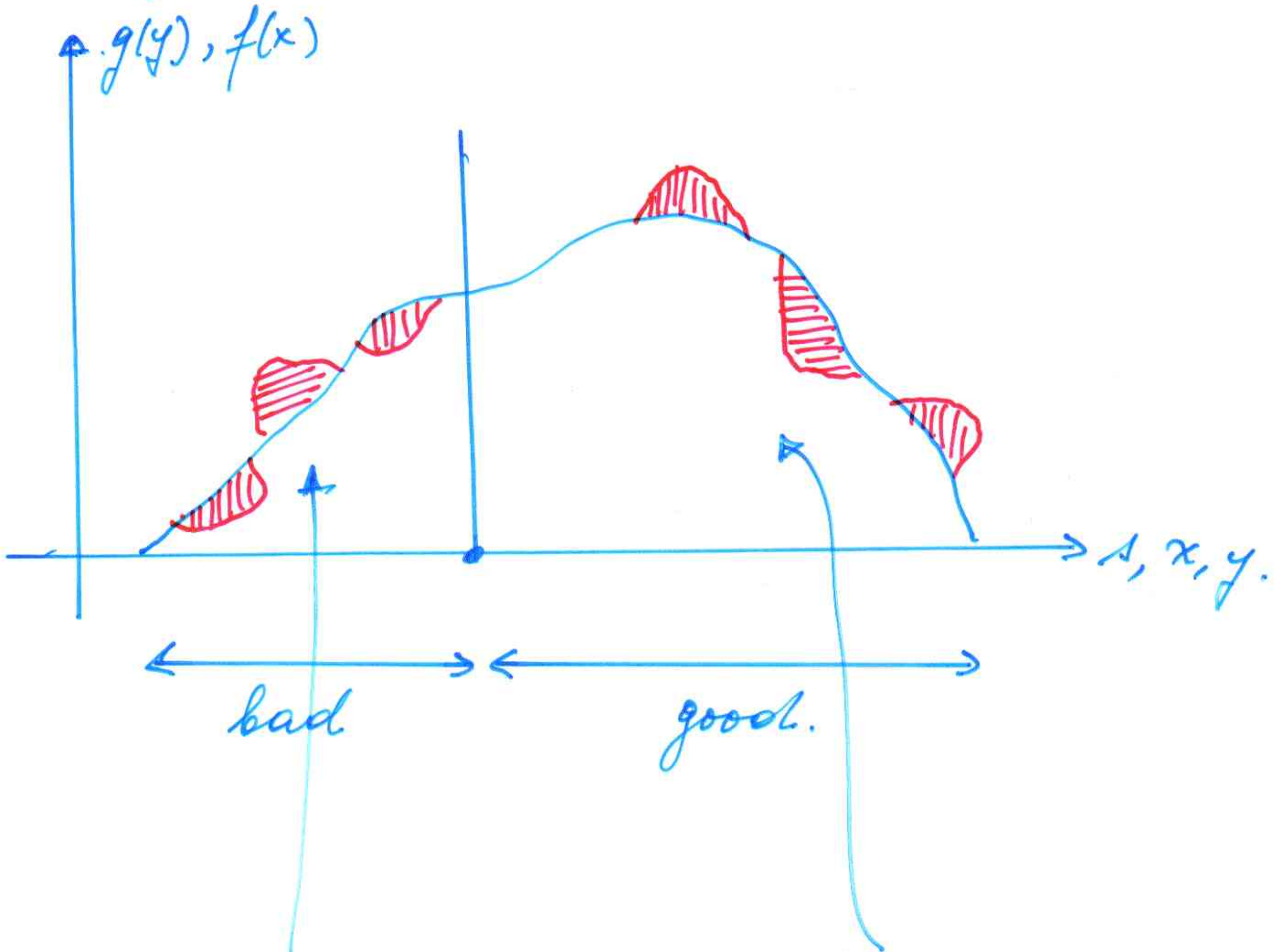
- risk aversion: from preference ($u'' < 0$) to decisions, using besides the DARA assumption

- prudence and temperance: from decisions to preferences (signs of u''' and $u^{(4)}$)

Hence: search for a preference based definition of prudence and temperance

I 2 A neglected paper

Geits, Menezes, Tressler, AER (1980)



MPC.

$u'' < 0 \Rightarrow$ welfare improving

MPS.

$u'' < 0 \Rightarrow$ welfare deteriorating.

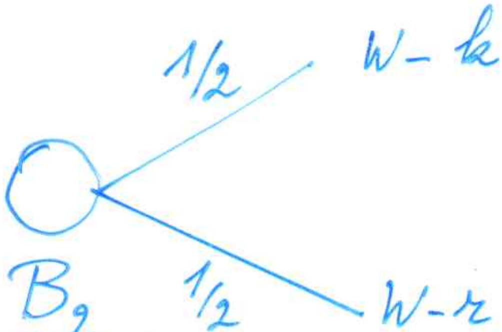
(Since the changes have the same "size" we face a Mean Variance Preserv. Transf.

$u''' > 0 \Leftrightarrow$ The change is globally beneficial

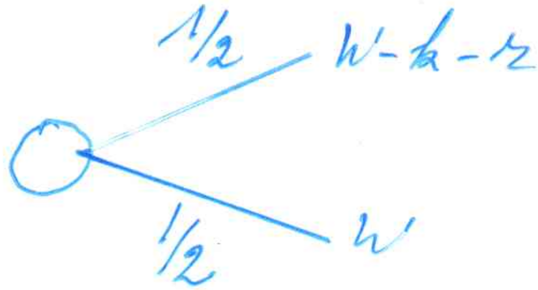
"

Downside risk aversion.

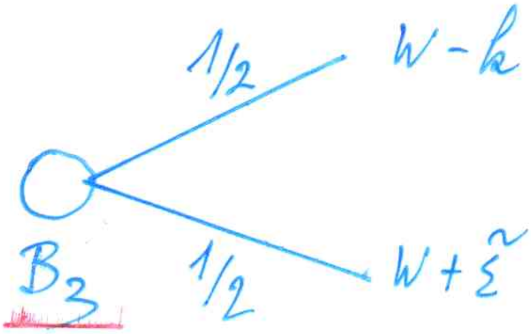
This is a preference approach in E.U.



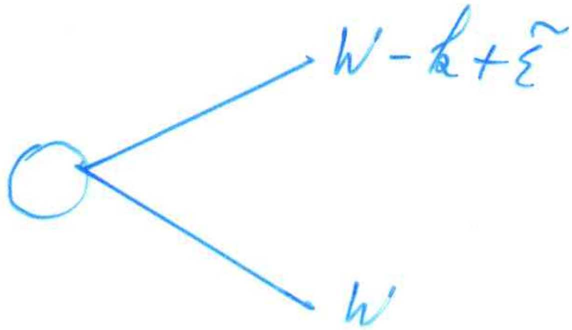
B_2



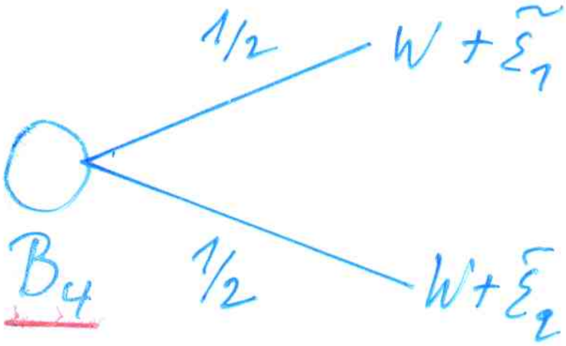
A_2



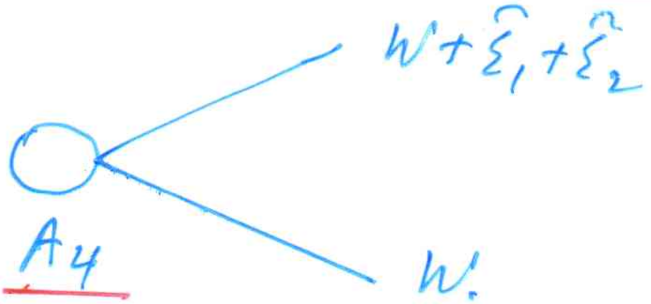
B_3



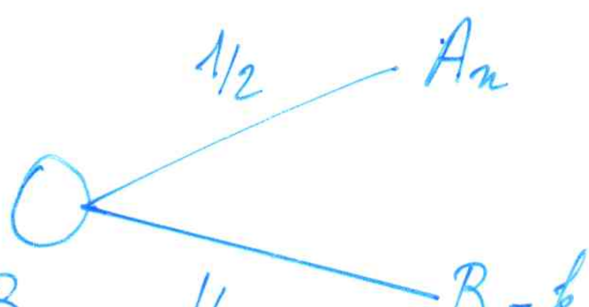
A_3



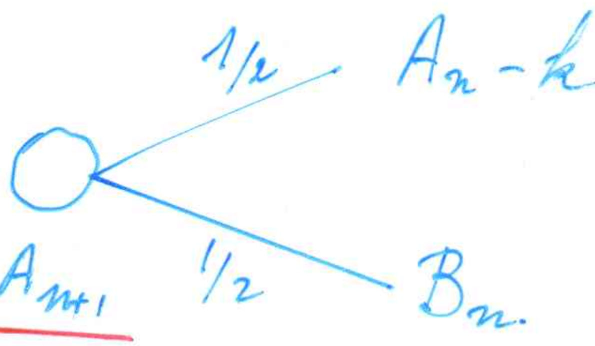
B_4



A_4



B_{n+1}



A_{n+1}

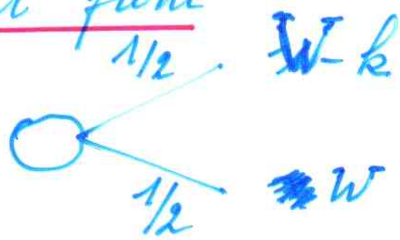
I 3 The notion of "risk apportionment" (pain)

Goal: try to extend GMT's contribution
- to higher orders.
- towards a "model free" approach.

Principle: - "I like to disaggregate pains"
- "Do not put all the pains on a single state of nature."

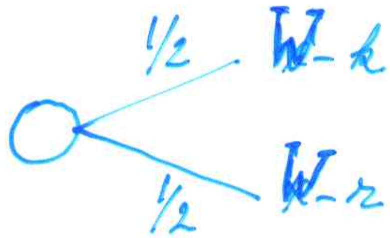
I Illustrations.

a) Start from

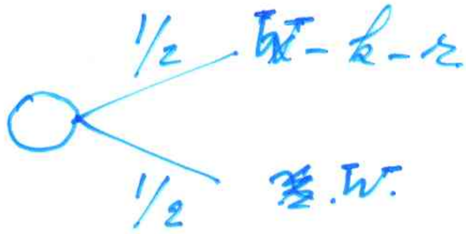


$\frac{1}{2} : -r.$

B_2



A_2



$$B_2 \succsim A_2.$$

Notice that
B_2 is a MPC of A_2. : we have defined risk aversion.

b) <u>Start from</u>

$$\bigcirc \begin{cases} \tfrac{1}{2} & W-k \\ \tfrac{1}{2} & W \end{cases} \qquad \frac{1}{2} : \tilde{\varepsilon} \text{ with } E(\tilde{\varepsilon}) = 0.$$

$$B_3 \ \bigcirc \begin{cases} \tfrac{1}{2} & W-k \\ \tfrac{1}{2} & W+\tilde{\varepsilon} \end{cases} \qquad A_3 \ \bigcirc \begin{cases} \tfrac{1}{2} & W-k+\tilde{\varepsilon} \\ \tfrac{1}{2} & W. \end{cases}$$

<u>$B_3 \succsim A_3$</u> will turn out to be <u>prudence</u>

<u>Remarks</u>

- Take $W = 2000$ $k = 1000$ $\tilde{\varepsilon} \ \bigcirc \begin{cases} \tfrac{1}{2} & -1000 \\ \tfrac{1}{2} & +1000. \end{cases}$
- Compare with Kimball's precaut. premium.

c) <u>Start from.</u>

$$\bigcirc \begin{cases} \tfrac{1}{2} & W+\tilde{\varepsilon}_1 \\ \tfrac{1}{2} & W \end{cases} \qquad \frac{1}{2} : \tilde{\varepsilon}_2 \perp \tilde{\varepsilon}_1 \quad E[\tilde{\varepsilon}_i] = 0.$$

$$B_4 \ \bigcirc \begin{cases} \tfrac{1}{2} & W+\tilde{\varepsilon}_1 \\ \tfrac{1}{2} & W+\tilde{\varepsilon}_2 \end{cases} \qquad A_4 \ \bigcirc \begin{cases} & W+\tilde{\varepsilon}_1+\tilde{\varepsilon}_2 \\ & W. \end{cases}$$

<u>$B_4 \succsim A_4$</u> will turn out to be <u>temperance.</u>

I 4. The translation in the E-U model

a) The tool:

the utility premium $(v(w))$

$$v(w) = E\left[u(w+\tilde{\varepsilon})\right] - u(w)$$

$$v(w) < 0 \iff u'' < 0.$$

Remark: compare with the risk premium (π)

$$E\left[u(w+\tilde{\varepsilon})\right] - u(w-\pi) = 0.$$

$$v(w) \approx \frac{\sigma^2}{2} u''(w) \qquad \pi \approx \frac{\sigma^2}{2}\left(-\frac{u''(w)}{u'(w)}\right)$$

$$\pi = \frac{-v(w)}{u'(w)}.$$

Properties.

$$v'(w) = E\left[u'(w+\tilde{\varepsilon})\right] - u'(w)$$

$$v'(w) > 0 \iff u''' > 0.$$

$$v''(w) = E\left[u''(w+\tilde{\varepsilon})\right] - u''(w)$$

$$v''(w) < 0 \iff u^{(4)} < 0.$$

By Jensen's inequality.

b) Link with fair apportionment.

$B_3 \succ A_3$ becomes in the E-U. model

$$\frac{1}{2} u(w-k) + \frac{1}{2} E\left[u(w+\tilde{\varepsilon})\right] > \frac{1}{2} E\left[u(w-k+\tilde{\varepsilon})\right] + \frac{1}{2} u(w)$$

$$u(w-k) - E\left[u(w-k+\tilde{\varepsilon})\right] > u(w) - E\left[u(w+\tilde{\varepsilon})\right]$$

$$v(w-k) > v(w)$$

$$v'(w) > 0 \iff u''' > 0.$$

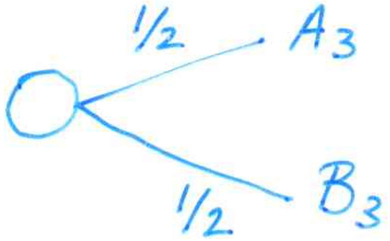
Conclusion: risk-aversion, prudence and temperance are all the consequences of a PREFERENCE expressed in a model-free setting ("I like to disaggregate pains").

II How to go to higher orders?

Illustration for $u^{(5)} > 0$ ("edginess")

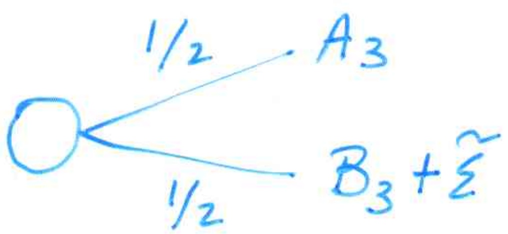
Two ways

a) **Start from**

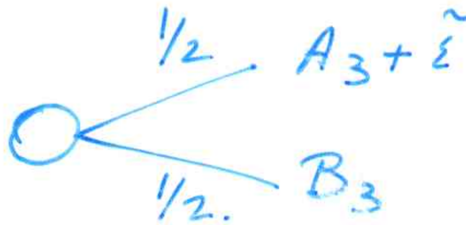


$\frac{1}{2} : \tilde{\varepsilon}$ with $E(\tilde{\varepsilon}) = 0$.

B_5

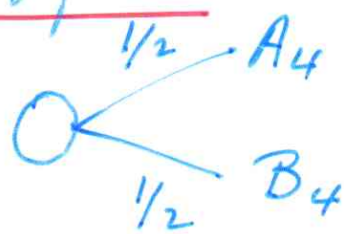


A_5



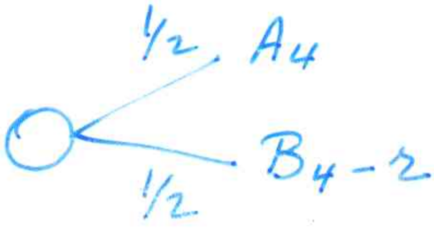
$B_5 \succsim A_5 \iff u^{(5)} > 0$. (in the E.U. model)

b) **Start from**

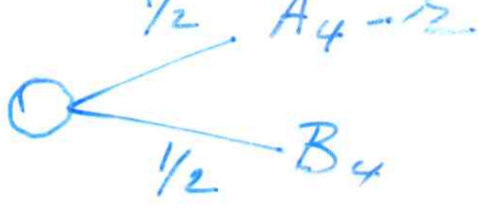


$\frac{1}{2} : -r$.

B_5'



A_5'



$B_5' \succsim A_5' \iff u^{(5)} > 0$ (in the E-U model).

Conclusion (at this stage)

The principle of preference for fair disaggregation justifies in E-U that the derivatives of U alternate in sign.

When a graduate student asked
"Why do you assume that $U^{(n)} > 0$"
the answer used to be.
"Because it gives sensible results"
Now you can say
"Because I like to disaggregate fair".

III Risk apportionment and stochastic dominance

Search for a more general principle

Model-free preference: "Combine good with bad" (return to G.M.T.)

Good and bad are defined from S.1.

Let:

$$\tilde{X}_N \succ_N \tilde{Y}_N. \qquad \tilde{X}_i \perp \tilde{X}_j \perp \tilde{Y}_i$$

$$\tilde{X}_M \succ_M \tilde{Y}_M. \qquad \perp \tilde{Y}_j$$

The principle

$$\left\{\tfrac{1}{2}\left(\tilde{X}_N+\tilde{Y}_M\right), \tfrac{1}{2}\left(\tilde{X}_M+\tilde{Y}_N\right)\right\}$$

$$\succ_? \left\{\tfrac{1}{2}\left(\tilde{X}_N+\tilde{X}_M\right), \tfrac{1}{2}\left(\tilde{Y}_N+\tilde{Y}_M\right)\right\}$$

The theorem.

$$? = N+M.$$

Notice that $(\tilde{X}_N+\tilde{X}_M) \succ (\tilde{X}_N+\tilde{Y}_M), (\tilde{X}_M+\tilde{Y}_N) \succ (\tilde{Y}_M+\tilde{Y}_M)$

The proof: very simple.

Tool: an extended version of the utility premium

$$v(w) = E\left[u(w+\tilde{Y})\right] - E\left[u(w+\tilde{X})\right]$$

where $\tilde{X} \succ \tilde{Y}$ so that $v < 0$.

Illustration

Risk aversion: $\tilde{X}_N = 1$, $\tilde{Y}_N = \tilde{X}_M = 0$.
$\tilde{Y}_M = -1$.

$1 \succ_1 0$.
$0 \succ_1 -1$. $\Rightarrow$ $0 \succ_2 \left\{ \frac{1}{2}(+1), \frac{1}{2}(-1) \right\}$.

Prudence. $\tilde{X}_N \succ_2 \tilde{Y}_N$.
$w \succ_1 w-k$.

$$\left\{ \frac{1}{2}\left(w-k+\tilde{X}_N\right), \frac{1}{2}\left(w+\tilde{Y}_N\right) \right\}$$

$$\succ_3 \left\{ \frac{1}{2}\left(w+\tilde{X}_N\right), \frac{1}{2}\left(w-k+\tilde{Y}_N\right) \right\}.$$

IV Extensions

IV.1. The multivariate utility function.

How to interpret the signs of successive cross derivatives of $u(x,y)$?

Consider first 2 sure pains:
$-k$ on x , $-r$ on y.

If I like to disaggregate pains

$$\left\{\tfrac{1}{2}(x-k,y), \tfrac{1}{2}(x,y-r)\right\} \succ \left\{\tfrac{1}{2}(x-k,y-r), \tfrac{1}{2}(x,y)\right\}.$$

Scott Richard shows in Mngt Sc 1975 that in the E-U model this preference implies

$u_{12} < 0.$

Extension to cross prudence, cross temperance. (E-R-S, Mngt Sc 2007).

cross prudence
pains: $\tilde{\varepsilon}$ on x $(E(\tilde{\varepsilon})=0)$
$-r$ on y.

cross temperance
pains: $\tilde{\varepsilon}_1$ on x $\qquad E(\tilde{\varepsilon}_i)=0$
$\tilde{\varepsilon}_2$ on y. $\qquad \tilde{\varepsilon}_1 \perp \tilde{\varepsilon}_2$

IV 2 Correlation aversion

Widespread idea : • hedging

• CAPM Mossin (1976): "... A rainwear manufacturer is worth relatively more if everybody else produces ice cream"

• solvency problems for insur. cies that insure correlated risks.

Let us consider an "elementary correlation increasing transformation"

(Epstein - Tanny, C. J. E., 1980)

Take 2 binary r.v. $\tilde{X}$, $\tilde{Y}$.

$\tilde{Y} \backslash \tilde{X}$	x_a	x_b	
y_a	$p_1 p_2 + e$	$p_1(1-p_2) - e$	p_1
y_b	$p_2(1-p_1) - e$	$(1-p_1)(1-p_2) + e$	$1-p_1$
	p_2	$1-p_2$	

$x_b > x_a$

$y_b > y_a$

$p_1 \leq p_2$

$e = 0$: independ.

$$-p_1 p_2 \leq e \leq p_1(1-p_2)$$

- Express $E[u]$
- Compute $\frac{dE[u]}{de} < 0$ by correl. aversion.
- Extend to higher orders of S.D. You recover all the previous results

IV 3. Non E-U models.

Question: what does the preference for fair disaggr. imply outside E-U?

. in Yaari's dual theory: does it give information on the distortions of probab.?

. in prospect theory. (risk loving behavior)

IV 4. Intensity

Prudence

- Kimball: intensity is measured by $-u'''/u''$

- Another measure: u'''/u'.

IV 5. Experimental Economics.

V Conclusion.

IV 6. Multiplicative risks.